

The University of Chicago

FOUNDED BY JOHN D. ROCKEFELLER.

On the
Reduction of Hyperelliptic Functions ($p=2$)
to
Elliptic Functions by a Transformation
of the Second Degree.

A Dissertation

Submitted to the Faculties of the Graduate Schools of Arts
Literature, and Science, in candidacy for the degree of

Doctor of Philosophy
(Department of Mathematics)

by

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Introduction.

The problem that is the subject of the following paper is not a new one. As early as 1832 Legendre¹⁾ had shown that the two hyperelliptic integrals

$$\int \frac{dx}{\sqrt{x(1-x^2)(1-\kappa^2 x^2)}} \quad \text{and} \quad \int \frac{x dx}{\sqrt{x(1-x^2)(1-\kappa^2 x^2)}}$$

are each expressible in terms of two elliptic integrals of the first kind by means of a quadratic transformation.

Immediately after, Jacobi²⁾ in a review of Legendre's work pointed out that this property belongs to two linearly independent integrals of a more general type than those discussed by Legendre, viz.

$$\int \frac{dx}{\sqrt{R(x)}} \quad \text{and} \quad \int \frac{x dx}{\sqrt{R(x)}}$$

where $R(x) = x(1-x)(1-k\lambda x)(1+kx)(1+\lambda x)$.

In 1867 Königsberger³⁾ proposed and solved the problem, to determine the necessary and sufficient condition on the roots of the sextic in order that the associated hyperelliptic integrals may be reducible to elliptic integrals by a transformation of the

1) Legendre — *Traité des Fonctions Elliptiques*, t. 3, pg. 333.

2) Jacobi — *Werke*, Bd. I, pg. 380.

3) Königsberger — *Crelle*, Bd. 67, pg. 58.

second degree. Denoting these roots by $a_1, \dots a_5, \infty$, he found the required condition to be

$$(a_1 - a_2)(a_1 - a_3) = (a_1 - a_4)(a_1 - a_5).$$

This relation expresses the fact that the roots are in involution, and shows that Jacobi's integrals are the most general integrals having the property proposed.

In obtaining the above result, Königsberger made use of a theorem communicated to him by Weierstrass to the effect that, whenever two hyperelliptic integrals of the first kind, linearly independent, can be found, which are reducible to elliptic integrals by an algebraic substitution it is always possible to transform the corresponding ϑ -functions into a new system such that the modulus $\tau'_{12} = 0$. In a later paper¹⁾ Königsberger gave a new proof of the same result, and extended to the problem of reducibility the algebraic transformation method employed by Jacobi with reference to elliptic integrals.

In 1869 Gordan²⁾, in connection with his investigations on the binary sextic, gave the algebraic condition for reducibility in invariant form. He showed that when the skew invariant of the sextic is zero, every associated hyperelliptic integral may be expressed as an aggregate of elliptic integrals by substituting as new variables the quadratic covariants l and m .

In 1876 the inversion problem was taken up by Hermite³⁾ who expressed the odd hyperelliptic functions belonging to the irrationality $\sqrt{R(x)}$ in terms of elliptic functions. Immediately following him Cayley⁴⁾ accomplished the like result for the even functions.

The solution of the inversion-problem is also contained implicitly in the work of Pringsheim⁵⁾, who, while studying the general quadratic transformation of the ϑ -functions, derived as a special case the formulae connecting the elliptic and hyperelliptic ϑ -functions, which he made use of to obtain the reduction of the integrals, previously effected by Jacobi by algebraic methods.

1) Königsberger — Ueber die Reduction hyperelliptische Integrale auf elliptische. Crelle Bd. 85, pg. 273.

2) Gordan — Applicazioni di alcuni risultati etc., Annali di Mathematica, Serie II. — Tomo II., pag. 346.

3) Hermite — Sur un exemple de réduction d'intégrales abéliennes, aux fonctions elliptiques. Annales de la Société scientifique de Bruxelles, t. I (1876).

4) Cayley — Comptes Rendus, t. 85, pp. 265, 426, 472 (1877).

5) Pringsheim — Mathematische Annalen, Bd. 9, pg. 445 (1875).

In 1882 Picard ¹⁾ proved that the relation among the $\mathfrak{S}$ -moduli previously found by Weierstrass may, by a proper transformation, be expressed in the form $\tau'_{12} = \frac{1}{k}$; and also that the reduction from the hyperelliptic to the elliptic integrals can be effected by an algebraic transformation of the k^{th} degree. This is a special case of a theorem announced a little later by Weierstrass ²⁾).

In the following pages I have attempted, at the suggestion and with the kind assistance of Professor Oskar Bolza, a new treatment of the reduction problem taking as my point of view the covariant character of the problem, first hinted at by Gordan. This suggests the systematic use of another normal form for the sextic, other than that employed by Jacobi, and adhered to by most of the subsequent writers, and the solution of the problem is greatly simplified thereby.

In the first part of the paper, the fact that the roots of the sextic form an involution is dwelt upon, and some theorems regarding involution, bearing directly on the subject in hand, are deduced. Besides the two reducible integrals usually mentioned, a complete list is given of those special cases of reducible integrals characterized by more than one involution on the roots of the sextic.

Following this, I enter upon a detailed examination of what I may call the general case of reducibility (one involution), beginning with a study of the correspondence between the hyperelliptic and elliptic Riemann surfaces, and the derivation of the reduction-formulae for integrals of the second and third kinds.

The remainder of the paper is chiefly devoted to a solution of the „inversion-problem“. Conforming to our point of view, the σ -functions introduced by Klein are used, and the three simple reduction-formulae of Theorem VI, are among the most important new results given in the present paper. They unite in an implicit form the results of both Hermite and Cayley, and contain even more, since the factors of proportionality which disappear when the quotients are formed are here preserved. The simplicity of these results and the ease with which they are derived clearly show the advantage of the present mode of treatment ³⁾.

1) Picard — Sur la réduction etc., Bulletin d. l. Soc. Math. d. France, t. 11, pg. 25.

2) Kowalevski — Acta Mathematica, Bd. 4, pg. 400 (1884).

3) Compare Cayley's remark: „Les reductions pour obtenir les valeurs des dix fonctions sent très-pénibles“. Comptes Rendus, t. 85, pg. 426.

Finally, the Weber and the Göpel relations satisfied by the σ -functions are derived, and attention is called to the way in which the Kummer surfaces, represented by them, specialize for the case of reducibility before us.

§ 1. *Algebraic Reduction by Jacobi's Method.*

Jacobi's method for the transformation of elliptic integrals ¹⁾, can be extended at once to the investigation of the reducibility of hyperelliptic integrals ²⁾ to elliptic integrals by a transformation of degree n . If $R(x_1, x_2)$ denote a binary sextic, and if

$$P(U, V) = (\kappa U + \kappa' V)(\lambda U + \lambda' V)(\mu U + \mu' V)(\nu U + \nu' V),$$

then this method, for $n = 2$, leads directly to the result:

Theorem I. *In order that the hyperelliptic integral of the first kind,*

$$\int \frac{(A_1 x_1 + A_2 x_2)(x_1 dx_2 - x_2 dx_1)}{\sqrt{R(x_1, x_2)}}$$

may be reducible to the elliptic integral

$$M \int \frac{UdV - VdU}{\sqrt{P(U, V)}},$$

M being a constant, by the quadratic transformation,

$$U = a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$$

$$V = b_0 x_1^2 + 2b_1 x_1 x_2 + b_2 x_2^2$$

it is necessary and sufficient that $R(x_1, x_2)$ be decomposable into three quadratic factors, φ, ψ, χ , such that

$$\begin{aligned} \varphi &= \kappa U + \kappa' V \\ \psi &= \lambda U + \lambda' V \\ \chi &= \mu U + \mu' V \end{aligned} \tag{1}$$

while the remaining factor of $P(U, V)$ must take the form,

$$\nu U + \nu' V = (A'_1 x_1 + A'_2 x_2)^2,$$

1) Fundamenta Nova, Werke, Bd. I, pg. 55.

2) Cf. Königsberger — Crelle, Bd. 85.

and besides, the condition must be satisfied,

$$(2) \quad M(UdV - VdU) = (A_1x_1 + A_2x_2)(A'_1x_1 + A'_2x_2)(x_1dx_2 - x_2dx_1).$$

By eliminating U and V from (1), we obtain a linear relation between φ , ψ , χ ,

$$\lambda_1\varphi + \lambda_2\psi + \lambda_3\chi = 0.$$

This is equivalent to the invariant condition $R = 0$, where R is the resultant of the three quadratic forms, φ , ψ , χ ¹⁾.

This condition, in turn, is equivalent to the vanishing of the skew invariant of the sextic²⁾.

Now, the two roots of $nU + n'V = 0$, regarded as an equation in $\frac{x_1}{x_2}$ with n , n' as variable parameters, represent pairs of points in an involution whose double-points are given by the quadratic equation

$$F(x_1, x_2) = \begin{vmatrix} \frac{\partial U}{\partial x_1} & \frac{\partial U}{\partial x_2} \\ \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{vmatrix} = 0.$$

Since

$$UdV - VdU = \frac{1}{2} F(x_1, x_2)(x_1dx_2 - x_2dx_1)$$

it follows from (2) that the numerator of the reducible hyperelliptic integral must contain one of the linear factors of $F(x_1, x_2)$. We reach, therefore, the conclusion:

Theorem II. *In order that*

$$\int \frac{(A_1x_1 + A_2x_2)(x dx)_3}{\sqrt{R(x_1, x_2)}}$$

may be reducible to an elliptic integral by a transformation of the second degree, it must be possible to decompose $R(x_1, x_2)$ into three quadratic factors, representing pairs of points in an involution, one of the double-points of which is given by the equation $A_1x_1 + A_2x_2 = 0$.

Conversely: If $R(x_1, x_2)$ can be decomposed into three quadratic factors belonging to an involution whose double-points are given by the equations,

$$(3) \quad A_1x_1 + A_2x_2 = 0, \quad A'_1x_1 + A'_2x_2 = 0,$$

1) Clebsch — Binäre Formen, pg. 208.

2) Salmon — Higher Algebra, 4th ed., pg. 270.

3) $(x dx)$ is written for brevity instead of the determinant, $(x_1dx_2 - x_2dx_1)$.

then there exist two reducible integrals, viz.,

$$(4) \quad \int \frac{(A_1 x_1 + A_2 x_2)(x dx)}{\sqrt{R(x_1, x_2)}}, \quad \int \frac{(A'_1 x_1 + A'_2 x_2)(x dx)}{\sqrt{R(x_1, x_2)}}.$$

The result just obtained admits a very simple geometrical interpretation. If we put $X_1 = x_1^2$, $X_2 = x_1 x_2$, $X_3 = x_2^2$, then to each set of values of x_1, x_2 corresponds a single point on the conic $X_1 X_3 - X_2^2 = 0$. To a pair of values of the ratio $\frac{x_1}{x_2}$ belonging to an involution, correspond the two points of intersection with the conic of one of the rays of a pencil whose center, P , does not lie on the curve¹⁾. The equations (3), then, represent the points of tangency of those two rays of the pencil which touch the curve.

§ 2. Canonical Form for the Reducible Integrals.

As is well known, three quadratic forms representing point-pairs in the same involution can be expressed as sums of squares by a simultaneous linear transformation²⁾. Accordingly, the transformation,

$$(5) \quad A_1 x_1 + A_2 x_2 = k z_1, \quad A'_1 x_1 + A'_2 x_2 = k' z_2,$$

where k, k' are arbitrary constants, enables us to put φ, ψ, χ in the form

$$(6) \quad \begin{aligned} \varphi &= \varphi_0 (z_1^2 - \alpha^2 z_2^2) \\ \psi &= \psi_0 (z_1^2 - \beta^2 z_2^2) \\ \chi &= \chi_0 (z_1^2 - \gamma^2 z_2^2) \end{aligned}$$

while the sextic $R(x_1, x_2)$ may, in consequence, be written in the form

$$(7) \quad f(z_1, z_2) = a_0 z_1^6 + 3a_1 z_1^4 z_2^2 + 3a_2 z_1^2 z_2^4 + a_3 z_2^6.$$

By the linear transformation (5), the integrals (4) are changed into such as differ by constant factors only from

$$(8) \quad \int \frac{z_1(z dz)}{\sqrt{f(z_1, z_2)}}, \quad \int \frac{z_2(z dz)}{\sqrt{f(z_1, z_2)}}.$$

1) See Reye — Geometrie der Lage, dritte Auflage, p. 139.

2) Clebsch — Binäre Formen, p. 208.

By the quadratic transformation $\zeta_1 = z_1^2$, $\zeta_2 = z_2^2$ we obtain

$$(9) \quad \begin{aligned} \int \frac{z_1(zdz)}{\sqrt{f(z_1, z_2)}} &= \int \frac{(\zeta d\zeta)}{\sqrt{R_1(\zeta_1, \zeta_2)}} \\ \int \frac{z_2(zdz)}{\sqrt{f(z_1, z_2)}} &= \int \frac{(\zeta d\zeta)}{\sqrt{R_2(\zeta_1, \zeta_2)}} \end{aligned}$$

where

$$(10) \quad \begin{aligned} R_1(\zeta_1, \zeta_2) &= 4\zeta_2(a_0\zeta_1^3 + 3a_1\zeta_1^2\zeta_2 + 3a_2\zeta_1\zeta_2^2 + a_3\zeta_2^3) \\ R_2(\zeta_1, \zeta_2) &= 4\zeta_1(a_0\zeta_1^3 + 3a_1\zeta_1^2\zeta_2 + 3a_2\zeta_1\zeta_2^2 + a_3\zeta_2^3). \end{aligned}$$

The normal form, (7), for the sextic, suggested as it is by the properties of the reducible integrals, offers decided advantages over the Richelot's normal form employed by Jacobi, and adhered to by Hermite, Cayley, and Königsberger, the Richelot's form having no special relation to the reduction problem before us. As already remarked in the introduction, one of the principle objects of the present paper is a simplification of the problem by a systematic use of this normal form¹⁾.

As a first example of this simplification, we are now in a position to establish as once a general theorem. For the greater convenience we pass to non-homogeneous notation.

Theorem III. *If $F(z, \sqrt{f(z)})$ denote any rational function of the two arguments, $z, \sqrt{f(z)}$, then the most general hyperelliptic integral, $\int F(z, \sqrt{f(z)}) dz$, is expressible in terms of two elliptic integrals, together with algebraic and logarithmic functions.*

For, it is evident that $F(z, \sqrt{f(z)})$ can be separated into the terms $G_1(z^2) + z \cdot G_2(z^2) + \sqrt{f(z)} \cdot H_1(z^2) + z \sqrt{f(z)} \cdot H_2(z^2)$, where G_1, G_2, H_1, H_2 are rational functions of z^2 . When we make the substitution $z^2 = \zeta$, and integrate, the first two terms lead to algebraic and logarithmic functions, while the last two are elliptic integrals.

By combining this theorem with the addition-theorem for elliptic integrals it follows at once that the sum

$$\int \frac{\sqrt{f(g)} dz}{(z-g) \sqrt{f(z)}} + \int \frac{\sqrt{f(g)} dz'}{(z+g) \sqrt{f(z')}}$$

is expressible in terms of two elliptic integrals together with

1) This normal form has already been employed by Picard (Bulletin de la Société Mathématique de France, t. 11 (1883), pg. 45), and by Bolza (Inaugural-Dissertation, Göttingen, 1886, p. 32).

logarithmic functions, a theorem obtained by Hermite by means of Abel's theorem ¹⁾).

The substitution (5) contains two arbitrary constants, k, k' . For our further purposes, however, it is convenient to determine these by the conditions,

$$(1) \quad \text{Determinant of transformation} = 1,$$

$$(2) \quad a_3 = a_0.$$

The coefficients in (7) are now *invariants* of the sextic, and we proceed to show how they may be expressed in terms of the simultaneous invariants of φ, ψ, χ . These are ²⁾

$$(11) \quad \begin{aligned} A_{\varphi\varphi} &= -2\varphi_0^2\alpha^2 & A_{\psi\chi} &= -\psi_0\chi_0(\beta^2 + \gamma^2) \\ A_{\psi\psi} &= -2\psi_0^2\beta^2 & A_{\chi\chi} &= -\chi_0^2\gamma^2(\gamma^2 + \alpha^2) \\ A_{\chi\chi} &= -2\chi_0^2\gamma^2 & A_{\varphi\psi} &= -\varphi_0\psi_0(\alpha^2 + \beta^2). \end{aligned}$$

The following useful combinations of these invariants are easily derived:

$$(12) \quad \begin{aligned} A_{\psi\chi}^2 - A_{\psi\psi}A_{\chi\chi} &= \psi_0^2\chi_0^2(\beta^2 - \gamma^2)^2 \\ A_{\chi\varphi}^2 - A_{\chi\chi}A_{\varphi\varphi} &= \chi_0^2\varphi_0^2(\gamma^2 - \alpha^2)^2 \\ A_{\varphi\psi}^2 - A_{\varphi\varphi}A_{\psi\psi} &= \varphi_0^2\psi_0^2(\alpha^2 - \beta^2)^2 \\ \\ A_{\varphi\varphi}A_{\psi\psi}A_{\chi\chi} &= 8a_0^2 \\ A_{\psi\chi}A_{\chi\varphi}A_{\varphi\psi} &= 9a_1a_2 - a_0^2 \\ (13) \quad (A_{\psi\psi}A_{\chi\chi} - A_{\psi\chi}^2)(A_{\chi\chi}A_{\varphi\varphi} - A_{\chi\varphi}^2)(A_{\varphi\varphi}A_{\psi\psi} - A_{\varphi\psi}^2) \\ &= 27[a_0^4 + 4a_0(a_1^3 + a_2^3) - 6a_0^2a_1a_2 - 3a_1^2a_2^2]. \end{aligned}$$

A word is perhaps necessary in regard to the derivation of this last equation. Comparing the left hand member with (12), we observe that it is the negative of the discriminant of the cubic form, $a_0\zeta_1^3 + 3a_1\zeta_1^2\zeta_2 + 3a_2\zeta_1\zeta_2^2 + a_0\zeta_2^3$. Referring to Clebsch's Binäre Formen, pg. 114, we see that it is equal to the bracketed expression in the right member of (13), except for a numerical

1) Annales d. l. Soc. Scien. d. Bruxelles, 1876, pg. 10.

2) The notation is the same as that used by Gordan, Invariantentheorie, Bd. II, p. 147.

factor. This factor is easily determined by assigning particular values to the roots, say, $\alpha^2 = i$, $\beta^2 = -i$, $\gamma^2 = -1$.

From the equations (13) we can obtain $a_0^2, a_1 a_2, a_0(a_1^3 + a_2^3)$ rationally in terms of $A_{\varphi\varphi}$, etc. Hence a_1^3, a_2^3 are the roots of a quadratic equation whose coefficients are rationally expressible in terms of $\sqrt{\frac{1}{2}A_{\varphi\varphi}}, \sqrt{\frac{1}{2}A_{\phi\phi}}, \sqrt{\frac{1}{2}A_{\chi\chi}}, A_{\phi\chi}, A_{\chi\varphi}, A_{\varphi\psi}$.

By combining equations (11) and (12) we obtain invariant expressions for $\varphi_0, \phi_0, \chi_0$, viz.,

$$(14) \quad \varphi_0 = \left[\frac{\sqrt{\frac{1}{2}A_{\varphi\varphi}}(A_{\phi\psi} + \sqrt{A_{\varphi\psi}^2 - A_{\phi\varphi}A_{\psi\psi}})(A_{\chi\varphi} - \sqrt{A_{\chi\varphi}^2 - A_{\chi\chi}A_{\varphi\varphi}})}{4\sqrt{\frac{1}{2}A_{\phi\phi}}\sqrt{\frac{1}{2}A_{\chi\chi}}} \right]^{\frac{1}{3}}$$

while ϕ_0, χ_0 are obtained directly from φ_0 by the cyclic permutations $(\varphi\psi\chi)$, $(\varphi\chi\psi)$ respectively. The ambiguity arising from the cube root is due to the separation of the sextic into three quadratic factors. The cube root of unity may be chosen arbitrarily for one of the functions, say φ_0 . The others are then determined. For, by eliminating $\alpha^2, \beta^2, \gamma^2$ from equations (11), the resulting equations in $\varphi_0, \phi_0, \chi_0$ involve only the ratios of these quantities. The signs of the radicals, $\sqrt{\frac{1}{2}A_{\varphi\varphi}}, \sqrt{\frac{1}{2}A_{\phi\phi}}, \sqrt{\frac{1}{2}A_{\chi\chi}}$, are entirely arbitrary. To determine the signs of the remaining radicals, we consider the three functional determinants of φ, ϕ, χ , which we will denote by $\vartheta_{\phi\chi}, \vartheta_{\chi\varphi}, \vartheta_{\varphi\phi}$. They satisfy the relation¹⁾

$$\varphi \vartheta_{\phi\chi} + \phi \vartheta_{\chi\varphi} + \chi \vartheta_{\varphi\phi} = 0.$$

We find for their explicit expressions

$$\vartheta_{\phi\chi} = \sqrt{A_{\phi\chi}^2 - A_{\phi\phi}A_{\chi\chi}} \cdot z_1 z_2$$

$$\vartheta_{\chi\varphi} = \sqrt{A_{\chi\varphi}^2 - A_{\chi\chi}A_{\varphi\varphi}} \cdot z_1 z_2$$

$$\vartheta_{\varphi\phi} = \sqrt{A_{\varphi\phi}^2 - A_{\varphi\varphi}A_{\phi\phi}} \cdot z_1 z_2$$

It is evident that the sign of one of the radicals may be arbitrary, while the signs of the others must be chosen so as to satisfy the above invariant relation.

If g_2, g_3 , and $\bar{g}_2, \bar{g}_3$ be the invariants of R_1 and R_2 respectively, these may be expressed directly in terms of the coefficients a_0, a_1, a_2^2 , and hence, by means of (13), in terms of $A_{\varphi\varphi}$, etc. Simil-

1) Clebsch — Binäre Formen, pg. 204.

2) See Klein — Theorie der Elliptische Modulfunctionen, Bd. I, pg. 15.

arly, if e_1, e_2, e_3 , and $\bar{e}_1, \bar{e}_2, \bar{e}_3$ be the roots of the two quartics, when put in the Weierstrass normal form, they may be expressed in terms of the roots, and hence in terms of the irrational invariants of the sextic as follows:

$$\begin{aligned}
 (15) \quad e_2 - e_3 &= a_0(\beta^2 - \gamma^2) = \varphi_0 E & \bar{e}_2 - \bar{e}_3 &= a_0\alpha^2(\beta^2 - \gamma^2) = \varphi_0\alpha^2 E \\
 e_3 - e_1 &= a_0(\gamma^2 - \alpha^2) = \psi_0 F & \bar{e}_3 - \bar{e}_1 &= a_0\beta^2(\gamma^2 - \alpha^2) = \psi_0\beta^2 F \\
 e_1 - e_2 &= a_0(\alpha^2 - \beta^2) = \chi_0 G & \bar{e}_1 - \bar{e}_2 &= a_0\gamma^2(\alpha^2 - \beta^2) = \chi_0\gamma^2 G \\
 e_1 + e_2 + e_3 &= 0 & \bar{e}_1 + \bar{e}_2 + \bar{e}_3 &= 0
 \end{aligned}$$

where

$$\begin{aligned}
 E &= \sqrt{A_{\psi\chi}^2 - A_{\psi\psi} A_{\chi\chi}} \\
 F &= \sqrt{A_{\chi\varphi}^2 - A_{\chi\chi} A_{\varphi\varphi}} \\
 G &= \sqrt{A_{\varphi\psi}^2 - A_{\varphi\varphi} A_{\psi\psi}}
 \end{aligned}$$

§ 3. *Special Cases in which there exist more than Two Reducible Integrals.*

It might happen that the roots of the fundamental sextic were in involution in more than one way. In such cases there would exist more than two hyperelliptic integrals of the first kind, each of which could be directly expressed by a single elliptic integral of the first kind. The determination of these can be reduced to the study of those sextics that admit linear transformations into themselves. For this purpose we make use of the following:

Lemma. A. *If three quadratic forms, having no common factors, whose discriminants are not zero, be in involution, then there exists one and but one binary collineation¹⁾ of period 2 which transforms the three simultaneously, each into itself, with a change of sign and an interchange of the roots in each.*

B. Conversely: *When three quadratic forms are simultaneously invariant for a binary collineation of period 2, which interchanges the roots of each, then the three forms are in involution.*

1) By a „binary collineation“ is meant a homogeneous linear transformation,

$$\begin{aligned}
 \rho z'_1 &= az_1 + bz_2 \\
 \rho z'_2 &= cz_1 + dz_2
 \end{aligned}$$

in which the factor of proportionality, ρ , is considered as non-essential, so that two substitutions differing only by the factor ρ are regarded as identical.

The truth of A appears immediately when we consider that the three forms may be expressed as the sums of squares by a linear transformation. Then, if the coefficients of the general linear transformation be determined on condition that it shall leave the three forms invariant, it will be found that the collineation

$$(16) \quad \begin{aligned} z'_1 &= iz_1 \\ z'_2 &= -iz_2 \end{aligned}$$

is the only one having the required property. It will also be found that this operation interchanges the roots, and reverses the sign of each form.

In order to prove B, let $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ be the three pairs of roots. There exists only one linear transformation that interchanges α and α' , β and β' . Written in the non-homogeneous form for convenience, it is

$$zz'(\alpha + \alpha' - \beta - \beta') + (z + z')(\beta\beta' - \alpha\alpha') + \alpha\alpha'(\beta + \beta') - \beta\beta'(\alpha + \alpha') = 0.$$

According to our assumption this equation must be satisfied by $z = \gamma, z' = \gamma'$, Hence

$$\gamma\gamma'(\alpha + \alpha' - \beta - \beta') + (\gamma + \gamma')(\beta\beta' - \alpha\alpha') + \alpha\alpha'(\beta + \beta') - \beta\beta'(\alpha + \alpha') = 0.$$

In determinant notation this relation may be written

$$\begin{vmatrix} 1 & \alpha + \alpha' & \alpha\alpha' \\ 1 & \beta + \beta' & \beta\beta' \\ 1 & \gamma + \gamma' & \gamma\gamma' \end{vmatrix} = 0,$$

which is the well-known condition for involution.

From this lemma follows: *To every decomposition of the sextic f into three quadratic factors, φ, ϕ, χ , belonging to an involution, there corresponds a collineation of period 2 which transforms f into itself at the same time interchanging its roots in pairs. And vice versa: To every collineation of period 2 which transforms f into itself, accompanied by an interchange of all the roots in pairs, corresponds a decomposition of f into three quadratic factors forming an involution.*

Hence, in order to solve the problem proposed at the beginning of this section, we have

(1) to determine all sextics with linear transformations into themselves, and

(2) to select from the automorphic transformations, belonging to each case, those of period 2, having the properties mentioned

in the lemma. To each such transformation correspond two reducible integrals.

The solution of the first part of this problem has already been given by Bolza in the American Journal, Vol. 10, pg. 50. The result there enunciated is:

Any binary sextic, f , with linear transformations into itself, (except of course the transformations, $z_1 = \pm iz'_1, z_2 = \pm iz'_2$), and whose roots are all distinct, is reducible by a linear substitution, of determinant 1, to one of the following canonical forms ¹⁾,

$$\text{I. } f = a_0 z_1^6 + 3a_1 z_1^4 z_2^2 + 3a_2 z_1^2 z_2^4 + a_0 z_2^6$$

(Cyclic group, $n = 2$)

$$\text{II. } f = a_0 z_1 (z_1^5 + z_2^5)$$

(Cyclic group, $n = 5$)

$$\text{III. } f = z_1 z_2 (a_0 z_1^4 + 2a_1 z_1^2 z_2^2 + a_0 z_2^4)$$

(Dihedron group, $n = 2$)

$$\text{IV. } f = a_0 z_1^6 + 2a_1 z_1^3 z_2^3 + a_0 z_2^6$$

(Dihedron group, $n = 3$)

$$\text{V. } f = a_0 (z_1^3 + z_2^3)$$

(Dihedron group, $n = 6$)

$$\text{VI. } f = a_0 z_1 z_2 (z_1^4 + z_2^4)$$

(Octahedron group).

The six roots of the sextic may be interpreted as points on the surface of a sphere of radius unity, and those substitutions that transform it into itself, as rotations of the sphere which leave the configuration of points as a whole unchanged ²⁾. We have, then, merely to select those rotations through the angle π , whose axes do not pass through any of the six points. To each such rotation corresponds a pair of reducible integrals whose numerators are represented by the two points on the sphere through which the axis of rotation passes. The total number of pairs of integrals for each case is respectively 1, 0, 2, 3, 4, 6. Case I is identical with the canonical form of § 2. For the remaining cases, the hyperelliptic, together with the equivalent elliptic integrals and the reduction formulae are given in the following tables.

1) The invariant criteria that characterize each case are given in the paper just referred to, pg. 70.

2) See Klein — Vorlesungen über das Ikosaeder, pg. 31.

Case III.

$\int \frac{(z_1 - z_2)(z dz)}{\sqrt{f(z_1, z_2)}}$	$\sqrt{\frac{2}{a_0 + a_1}} \int \frac{(\zeta d\zeta)}{\sqrt{\zeta_2(\zeta_2 - \zeta_1)[\zeta_1^2 + \frac{6a_0 - 2a_1}{a_0 + a_1} \zeta_1 \zeta_2 + \zeta_2^2]}}$	$\zeta_1 = (z_1 - z_2)^2$ $\zeta_2 = (z_1 + z_2)^2$
$\int \frac{(z_1 + z_2)(z dz)}{\sqrt{f(z_1, z_2)}}$	$\sqrt{\frac{2}{a_0 + a_1}} \int \frac{(\zeta d\zeta)}{\sqrt{\zeta_1(\zeta_2 - \zeta_1)[\zeta_1^2 + \frac{6a_0 - 2a_1}{a_0 + a_1} \zeta_1 \zeta_2 + \zeta_2^2]}}$	
$\int \frac{(z_1 - iz_2)(z dz)}{\sqrt{f(z_1, z_2)}}$	$\sqrt{\frac{2}{i(a_0 - a_1)}} \int \frac{(\zeta d\zeta)}{\sqrt{\zeta_2(\zeta_2 - \zeta_1)[\zeta_1^2 + \frac{6a_0 + 2a_1}{a_0 - a_1} \zeta_1 \zeta_2 + \zeta_2^2]}}$	$\zeta_1 = (z_1 - iz_2)^2$ $\zeta_2 = (z_1 + iz_2)^2$
$\int \frac{(z_1 + iz_2)(z dz)}{\sqrt{f(z_1, z_2)}}$	$\sqrt{\frac{2}{i(a_0 - a_1)}} \int \frac{(\zeta d\zeta)}{\sqrt{\zeta_1(\zeta_2 - \zeta_1)[\zeta_1^2 + \frac{6a_0 + 2a_1}{a_0 - a_1} \zeta_1 \zeta_2 + \zeta_2^2]}}$	

Case IV.

$\int \frac{(\varrho_1 - \varrho_2)(\varepsilon d\varrho)}{\sqrt{f(\varrho_1, \varrho_2)}}$ $\int \frac{(\varrho_1 + \varrho_2)(\varepsilon d\varrho)}{\sqrt{f(\varrho_1, \varrho_2)}}$	$\int \frac{\sqrt{2}(\zeta d\zeta)}{\sqrt{\zeta_2[(a_0 - a_1)\zeta_1^3 + (15a_0 + 3a_1)\zeta_1^2\zeta_2 + (15a_0 - 3a_1)\zeta_1\zeta_2^2 + (a_0 + a_1)\zeta_2^3]}}$ $\int \frac{\sqrt{2}(\zeta d\zeta)}{\sqrt{\zeta_1[(a_0 - a_1)\zeta_1^3 + (15a_0 + 3a_1)\zeta_1^2\zeta_2 + (15a_0 - 3a_1)\zeta_1\zeta_2^2 + (a_0 + a_1)\zeta_2^3]}}$	$\zeta_1 = (\varrho_1 - \varrho_2)^2$ $\zeta_2 = (\varrho_1 + \varrho_2)^2$
$\int \frac{(\varrho_1 - \varepsilon \varrho_2)(\varepsilon d\varrho)}{\sqrt{f(\varrho_1, \varrho_2)}}$ $\int \frac{(\varrho_1 + \varepsilon \varrho_2)(\varepsilon d\varrho)}{\sqrt{f(\varrho_1, \varrho_2)}}$	$\int \frac{\varepsilon^2 \sqrt{2}(\zeta d\zeta)}{\sqrt{\zeta_2[(a_0 - a_1)\zeta_1^3 + (15a_0 + 3a_1)\zeta_1^2\zeta_2 + (15a_0 - 3a_1)\zeta_1\zeta_2^2 + (a_0 + a_1)\zeta_2^3]}}$ $\int \frac{\varepsilon^2 \sqrt{2}(\zeta d\zeta)}{\sqrt{\zeta_1[(a_0 - a_1)\zeta_1^3 + (15a_0 + 3a_1)\zeta_1^2\zeta_2 + (15a_0 - 3a_1)\zeta_1\zeta_2^2 + (a_0 + a_1)\zeta_2^3]}}$	$\zeta_1 = (\varrho_1 - \varepsilon \varrho_2)^2$ $\zeta_2 = (\varrho_1 + \varepsilon \varrho_2)^2$ $\varepsilon = \frac{2\pi i}{3} \varrho$
$\int \frac{(\varrho_1 - \varepsilon^2 \varrho_2)(\varepsilon d\varrho)}{\sqrt{f(\varrho_1, \varrho_2)}}$ $\int \frac{(\varrho_1 + \varepsilon^2 \varrho_2)(\varepsilon d\varrho)}{\sqrt{f(\varrho_1, \varrho_2)}}$	$\int \frac{\varepsilon \sqrt{2}(\zeta d\zeta)}{\sqrt{\zeta_2[(a_0 - a_1)\zeta_1^3 + (15a_0 + 3a_1)\zeta_1^2\zeta_2 + (15a_0 - 3a_1)\zeta_1\zeta_2^2 + (a_0 + a_1)\zeta_2^3]}}$ $\int \frac{\varepsilon \sqrt{2}(\zeta d\zeta)}{\sqrt{\zeta_1[(a_0 - a_1)\zeta_1^3 + (15a_0 + 3a_1)\zeta_1^2\zeta_2 + (15a_0 - 3a_1)\zeta_1\zeta_2^2 + (a_0 + a_1)\zeta_2^3]}}$	$\zeta_1 = (\varrho_1 - \varepsilon^2 \varrho_2)^2$ $\zeta_2 = (\varrho_1 + \varepsilon^2 \varrho_2)^2$

Case V.

$\int \frac{z_1(zdz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{z_2(zdz)}{\sqrt{f(z_1, z_2)}}$	$\frac{1}{2} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_2 (\zeta_1^3 + \zeta_2^3)}}$ $\frac{1}{2} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_1^3 + \zeta_2^3)}}$	$\zeta_1 = z_1^3$ $\zeta_2 = z_2^3$
$\int \frac{(z_1 - z_2)(zdz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{(z_1 + z_2)(zdz)}{\sqrt{f(z_1, z_2)}}$	$\int \frac{\sqrt{2} (\zeta d\zeta)}{\sqrt{a_0 \zeta_2 (\zeta_1 + \zeta_2) (\zeta_1^2 + 14 \zeta_1 \zeta_2 + \zeta_2^2)}}$ $\int \frac{\sqrt{2} (\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_1 + \zeta_2) (\zeta_1^2 + 14 \zeta_1 \zeta_2 + \zeta_2^2)}}$	$\zeta_1 = (z_1 - z_2)^2$ $\zeta_2 = (z_1 + z_2)^2$
$\int \frac{(z_1 - \rho z_2)(zdz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{(z_1 + \rho z_2)(zdz)}{\sqrt{f(z_1, z_2)}}$	$\frac{\sqrt{2}}{\rho} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_2 (\zeta_1 + \zeta_2) (\zeta_1^2 + 14 \zeta_1 \zeta_2 + \zeta_2^2)}}$ $\frac{\sqrt{2}}{\rho} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_1 + \zeta_2) (\zeta_1^2 + 14 \zeta_1 \zeta_2 + \zeta_2^2)}}$	$\zeta_1 = (z_1 - \rho z_2)^2$ $\zeta_2 = (z_1 + \rho z_2)^2$ $\rho = e^{\frac{2\pi i}{6}}$
$\int \frac{(z_1 - \rho^2 z_2)(zdz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{(z_1 + \rho^2 z_2)(zdz)}{\sqrt{f(z_1, z_2)}}$	$\frac{\sqrt{2}}{\rho^2} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_2 (\zeta_1 + \zeta_2) (\zeta_1^2 + 14 \zeta_1 \zeta_2 + \zeta_2^2)}}$ $\frac{\sqrt{2}}{\rho^2} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_1 + \zeta_2) (\zeta_1^2 + 14 \zeta_1 \zeta_2 + \zeta_2^2)}}$	$\zeta_1 = (z_1 - \rho^2 z_2)^2$ $\zeta_2 = (z_1 + \rho^2 z_2)^2$

Case VI.

$\int \frac{(z_1 - z_2)(z dz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{(z_1 + z_2)(z dz)}{\sqrt{f(z_1, z_2)}}$	$\int \frac{\sqrt{2}(\zeta d\zeta)}{\sqrt{a_0 \zeta_2 (\zeta_2 - \zeta_1) (\zeta_1^2 + 6 \zeta_1 \zeta_2 + \zeta_2^2)}}$ $\int \frac{\sqrt{2}(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_2 - \zeta_1) (\zeta_1^2 + 6 \zeta_1 \zeta_2 + \zeta_2^2)}}$	$\zeta_1 = (z_1 - z_2)^2$ $\zeta_2 = (z_1 + z_2)^2$
$\int \frac{(z_1 - iz_2)(z dz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{(z_1 + iz_2)(z dz)}{\sqrt{f(z_1, z_2)}}$	$\sqrt{\frac{2}{i}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_2 - \zeta_1) (\zeta_1^2 + 6 \zeta_1 \zeta_2 + \zeta_2^2)}}$ $\sqrt{\frac{2}{i}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_2 - \zeta_1) (\zeta_1^2 + 6 \zeta_1 \zeta_2 + \zeta_2^2)}}$	$\zeta_1 = (z_1 - iz_2)^2$ $\zeta_2 = (z_1 + iz_2)^2$
$\int \frac{[z_1 + j(1 - \sqrt{2})z_2](z dz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{[z_1 + j(1 + \sqrt{2})z_2](z dz)}{\sqrt{f(z_1, z_2)}}$	$\sqrt{\frac{c}{j}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_2 (\zeta_2 - \zeta_1) (\zeta_1 + d\zeta_1) (\zeta_1 - d^2 \zeta_2)}}$ $\sqrt{\frac{c}{j}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_2 - \zeta_1) (\zeta_1 + d\zeta_1) (\zeta_1 - d^2 \zeta_2)}}$	$\zeta_1 = [z_1 + j(1 - \sqrt{2})z_2]^2$ $\zeta_2 = [z_1 + j(1 + \sqrt{2})z_2]^2$ $j = \frac{2\sqrt{-1}}{c}$ $c = 20 - 14\sqrt{2}$ $d = 3 - 2\sqrt{2}$

$\int \frac{[z_1 - j(1 - \sqrt{2})z_2](zdz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{[z_1 - j(1 + \sqrt{2})z_2](zdz)}{\sqrt{f(z_1, z_2)}}$	$\sqrt{\frac{c}{-j}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_2 (\zeta_2 - \zeta_1) (\zeta_1 + d\zeta_2) (\zeta_1 - d^2 \zeta_2)}}$ $\sqrt{\frac{c}{-j}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_2 - \zeta_1) (\zeta_1 + d\zeta_2) (\zeta_1 - d^2 \zeta_2)}}$	$\zeta_1 = [z_1 - j(1 - \sqrt{2})z_2]^2$ $\zeta_2 = [z_1 - j(1 + \sqrt{2})z_2]^2$
$\int \frac{[z_1 + j^3(1 - \sqrt{2})z_2](zdz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{[z_1 + j^3(1 + \sqrt{2})z_2](zdz)}{\sqrt{f(z_1, z_2)}}$	$\sqrt{\frac{c}{j^3}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_2 (\zeta_2 - \zeta_1) (\zeta_1 + d\zeta_2) (\zeta_1 - d^2 \zeta_2)}}$ $\sqrt{\frac{c}{j^3}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_2 - \zeta_1) (\zeta_1 + d\zeta_2) (\zeta_1 - d^2 \zeta_2)}}$	$\zeta_1 = [z_1 + j^3(1 - \sqrt{2})z_2]^2$ $\zeta_2 = [z_1 + j^3(1 + \sqrt{2})z_2]^2$
$\int \frac{[z_1 - j^3(1 - \sqrt{2})z_2](zdz)}{\sqrt{f(z_1, z_2)}}$ $\int \frac{[z_1 - j^3(1 + \sqrt{2})z_2](zdz)}{\sqrt{f(z_1, z_2)}}$	$\sqrt{\frac{c}{-j^3}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_2 (\zeta_2 - \zeta_1) (\zeta_1 + d\zeta_2) (\zeta_1 - d^2 \zeta_2)}}$ $\sqrt{\frac{c}{-j^3}} \int \frac{(\zeta d\zeta)}{\sqrt{a_0 \zeta_1 (\zeta_2 - \zeta_1) (\zeta_1 + d\zeta_2) (\zeta_1 - d^2 \zeta_2)}}$	$\zeta_1 = [z_1 - j^3(1 - \sqrt{2})z_2]^2$ $\zeta_2 = [z_1 - j^3(1 + \sqrt{2})z_2]^2$

The involution character of the different cases is well exhibited geometrically¹⁾, when we represent the binary field, $z_1:z_2$, as the totality of points situated on a conic. If the six points representing the roots of the sextic are in involution, it is possible to join them in pairs by three lines which meet in a point.

A collineation of period 2 may transform the sextic into itself as

(a) 3 point-pairs (Fig. I)

(b) 2 double-points, and
2 point-pairs. (Fig. II).

A collineation of kind (a) interchanges all the roots of the sextic, and hence leads to reducible integrals; while a collineation of kind (b) evidently does not.

The above cases may be illustrated by the accompanying figures, where, for the sake of convenience, the conic is metrically specialized, but, projectively considered, it is to be regarded as general. We observe, further, that the roots of the sextic are not necessarily real.

III. Two involutions²⁾,

$\frac{ABC}{DEF}, \frac{ABC}{DFE}$, (Fig. III).

IV. The sextic consists of the six intersections of a circle with an equilateral triangle, whose center is the same as that of the circle.

Fig. I

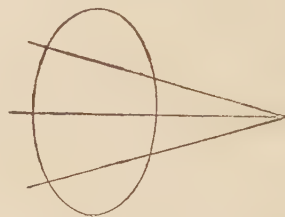
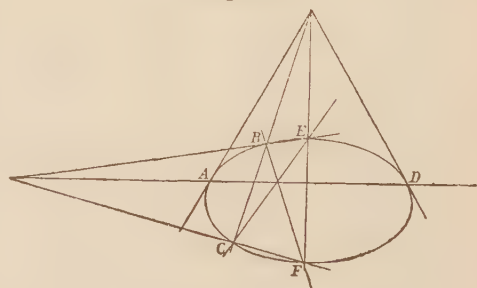


Fig. II



Fig. III



1) I am indebted to Prof. E. H. Moore for what follows in reference to this topic.

2) The notation, $\frac{ABC}{DEF}$, means that A and D, B and E, C and F are paired in the involution.

Three involutions, $\begin{smallmatrix} ABC \\ DFE \end{smallmatrix}$,
 $\begin{smallmatrix} ABC \\ FED \end{smallmatrix}$, $\begin{smallmatrix} ABC \\ EDF \end{smallmatrix}$, (Fig. IV).

V. The sextic consists of the six vertices of a regular hexagon inscribed in a circle.

Four involutions $\begin{smallmatrix} ABC \\ DEF \end{smallmatrix}$,
 $\begin{smallmatrix} ABC \\ FED \end{smallmatrix}$, $\begin{smallmatrix} ABC \\ EDF \end{smallmatrix}$, $\begin{smallmatrix} ABC \\ DFE \end{smallmatrix}$, (Fig. V).

VI. The sextic consists of three mutually harmonic point-pairs, and is represented by the vertices of a square inscribed in a circle, together with the two imaginary circular points at infinity, C_∞ , F_∞ .

Six involutions, $\begin{smallmatrix} ABE \\ DCF \end{smallmatrix}$,
 $\begin{smallmatrix} ABC \\ DFE \end{smallmatrix}$, $\begin{smallmatrix} BAD \\ ECF \end{smallmatrix}$, $\begin{smallmatrix} BAD \\ EFC \end{smallmatrix}$, $\begin{smallmatrix} CAD \\ FEB \end{smallmatrix}$, $\begin{smallmatrix} CAD \\ FBE \end{smallmatrix}$, (Fig. VI).

Fig. IV

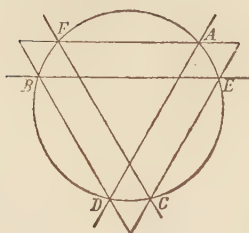


Fig. V

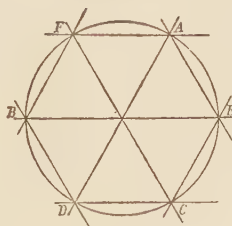
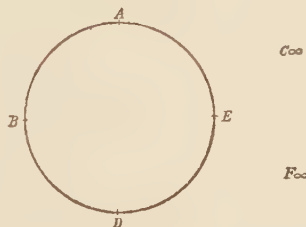


Fig. VI



§ 4. Periods of the Integrals of the First Kind.

Introducing non-homogeneous notation, we proceed to a more detailed consideration of Case I, beginning with a study of the periods of the integrals,

$$w_1 = \int \frac{z dz}{\sqrt{f(z)}}, \quad w_2 = \int \frac{dz}{\sqrt{f(z)}},$$

and their corresponding paths in the Riemann surface represented by the function

$$(19) \quad \sqrt{f(z)} = \sqrt{a_0 z^6 + 3a_1 z^4 + 3a_2 z^2 + a_0} = \sqrt{a_0 (z^2 - \alpha^2)(z^2 - \beta^2)(z^2 - \gamma^2)}.$$

$$(20) \quad \begin{aligned} \text{If } R_1(\zeta) &= 4(a_0 \zeta^3 + 3a_1 \zeta^2 + 3a_2 \zeta + a_0) = 4a_0(\zeta - \alpha^2)(\zeta - \beta^2)(\zeta - \gamma^2) \\ R_2(\zeta) &= 4\zeta(a_0 \zeta^3 + 3a_1 \zeta^2 + 3a_2 \zeta + a_0) = 4a_0 \zeta(\zeta - \alpha^2)(\zeta - \beta^2)(\zeta - \gamma^2) \end{aligned}$$

then, by the transformations

$$(21) \quad (a) \left\{ \begin{array}{l} \zeta = z^2 \\ \sqrt{R_1(\zeta)} = 2\sqrt{f(z)}, \end{array} \right. \quad (b) \left\{ \begin{array}{l} \zeta = z^2 \\ \sqrt{R_2(\zeta)} = 2z\sqrt{f(z)} \end{array} \right.$$

we have

$$(22) \quad \int \frac{z dz}{\sqrt{f(z)}} = \int \frac{d\zeta}{\sqrt{R_1(\zeta)}}, \quad \int \frac{dz}{\sqrt{f(z)}} = \int \frac{d\zeta}{\sqrt{R_2(\zeta)}}.$$

By these two transformations, the hyperelliptic Riemann surface $\mathbb{T}$ is brought into a two to one correspondence with each of the elliptic surfaces belonging to the functions, $\sqrt{R_1(\zeta)}$ and $\sqrt{R_2(\zeta)}$, which may be designated respectively by $\mathbb{T}$ and $\bar{\mathbb{T}}$. Lines radiating from the origin (i. e. the point $z = 0$), and concentric circles about the origin in P correspond to similar paths in $\mathbb{T}$ and $\bar{\mathbb{T}}$.

Fig. VII

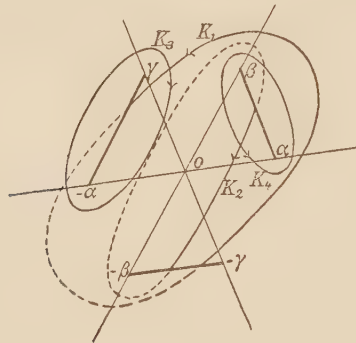


Fig. VIII

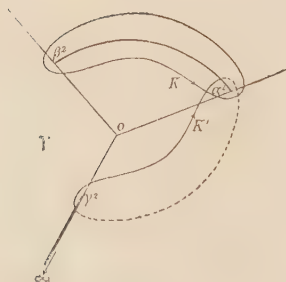
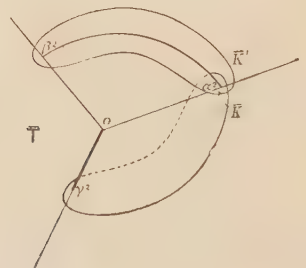


Fig. IX



Let the two-leaved surface, T (Fig. VII), be regarded as cut into pieces along the straight lines joining the origin with the branch-points, α, β, γ , etc., and extending to infinity. In the elliptic surface $\mathbb{T}$, the branch-points are $\alpha^2, \beta^2, \gamma^2, \infty$. We choose the branch-lines so as to join α^2 to β^2 , and γ^2 to ∞ . If the branch-line $\alpha^2\beta^2$ be the conform representation of the line $\alpha\beta$ in T , and if, further, we agree that the point at the origin in the upper sheet of T shall correspond to the like point in $\mathbb{T}$, Then the strip $\alpha^2 0 \beta^2$, extending to infinity, corresponds exactly with the strip $\alpha 0 \beta$, upper sheet with upper, and lower with lower. A similar correspondence holds between the strips $\beta^2 0 \gamma^2$ and $\beta 0 \gamma$.

The upper sheet of $\gamma^2 0 \alpha^2$ corresponds to the strip between $\gamma 0 - \alpha$, beginning with the upper sheet at 0 and passing into the lower sheet beyond the branch-line. The remaining sheet of $\gamma 0 - \alpha$ corresponds to the lower sheet of $\gamma^2 - 0 \alpha^2$. In a similar manner, the upper sheet of $-\alpha 0 - \beta$ corresponds to the continuous sheet in $\alpha^2 0 \beta^2$ that lies above from 0 to the branch-line, and below beyond it. The remaining sheets then correspond. The upper sheet of $\beta^2 0 \gamma^2$ corresponds to the sheet of $-\beta 0 - \gamma$ that is above from 0 to the branch-line, and below beyond it. The remaining sheets then correspond. Finally, the strip $-\gamma 0 \alpha$ corresponds to $\gamma^2 0 \alpha^2$ without any inversion in the order of the sheets.

In the surface $\bar{\mathbb{T}}$, the branch-point 0 replaces ∞ of $\mathbb{T}$, and, accordingly, the second branch-line runs from 0 to γ^2 . As in the previous case, the strips $\alpha 0 \beta$, $\beta 0 \gamma$, $-\gamma 0 \alpha$ correspond respectively to $\alpha^2 0 \beta^2$, $\beta^2 0 \gamma^2$, $\gamma^2 0 \alpha^2$. But that sheet of $\gamma 0 - \alpha$ which is above at 0, and below beyond the branch-line, corresponds to the lower sheet of $\gamma^2 0 \alpha^2$. The upper sheet of $-\alpha 0 - \beta$ corresponds to that which is below at 0, and above beyond the branch-line in the strip $\alpha^2 0 \beta^2$; while a similar correspondence of sheets holds in regard to the strips $-\beta 0 - \gamma$ and $\beta^2 0 \gamma^2$.

Having established this correspondence between the hyperelliptic, and the two elliptic Riemann surfaces, it is easily seen that to a path in the former corresponds unambiguously a certain path in either of the latter; but to a path in an elliptic surface corresponds simultaneously two paths in the hyperelliptic.

Of particular interest are the period-paths. In Fig. VII is represented a canonical system of such paths, so chosen as to represent the reduction properties of the periods in the simplest form. The images of K_1, K_2, K_3 may be made to coincide with $2K$ (i. e. K twice traversed), K', K respectively, by simple deformations which respect the connectivity of the surface, while

the image of K_2 can be contracted into a point. Likewise, in $\bar{\Gamma}$ the images of K_2, K_3, K_4 can be made to coincide with $2\bar{K}, \bar{K}, \bar{K}'$, while the path corresponding to K_1 can be contracted to a point.

By means of these correspondences we may at once express the hyperelliptic periods in terms of the elliptic. Let $2\omega, 2\omega'$ be the values obtained by integrating $u_1 = \int \frac{d\zeta}{\sqrt{R_1(\zeta)}}$ along the paths K and K' respectively. Also let $2\bar{\omega}, 2\bar{\omega}'$ be similar results for the integral $\bar{u}_1 = \int \frac{d\bar{\zeta}}{\sqrt{\bar{R}_2(\bar{\zeta})}}$, when referred to the paths $\bar{K}, \bar{K}'$. Then we find for w_1, w_2 the following table of periods:

	K_1	K_2	K_3	K_4
w_1	4ω	0	$2\omega'$	2ω
w_2	0	$4\bar{\omega}$	$2\bar{\omega}$	$2\bar{\omega}'$

If we put

$$v_1 = \frac{1}{4\omega} w_1,$$

$$v_2 = \frac{1}{4\bar{\omega}} w_2,$$

then v_1, v_2 are normal integrals of the first kind with the table of periods,

	K_1	K_2	K_3	K_4
v_1	1	0	$\frac{1}{2} \frac{\omega'}{\bar{\omega}}$	$\frac{1}{2}$
v_2	0	1	$\frac{1}{2}$	$\frac{1}{2} \frac{\bar{\omega}'}{\omega}$

This table establishes the truth of the Picard-Weierstrass theorem¹⁾ for our case, viz., $\tau_{12} = \frac{1}{2}$.

§ 5. Integrals of the Second and Third Kind.

In the present section we propose to discuss the reduction to elliptic integrals of Klein's commutative integral²⁾ of the third kind

$$Q_{xy}^{x'y'} = \int_y^x \int_{y'}^{x'} \frac{\sqrt{f(z_1, z_2)} \sqrt{f(z'_1, z'_2)} + a_z^3 a_{z'}}{2(z z')} \cdot \frac{(z dz)(z' dz')}{\sqrt{f(z_1, z_2)} \sqrt{f(z'_1, z'_2)}},$$

and Weierstrass' „Normal integrals of the second kind“, w_3, w_4 , associated³⁾ with the two reducible integrals of the first kind and with the integral $Q_{xy}^{x'y'}$.

A. Integrals of the second kind. The explicit expressions for w_3, w_4 are given by Wiltheiss (l. c. pg. 276). For our canonical form (7) they are

$$w_3 = -\frac{1}{10} \int \frac{(5a_0 z_1^3 + 3a_1 z_1 z_2^2)(z dz)}{z_2^2 \sqrt{f(z_1, z_2)}},$$

$$w_4 = -\frac{1}{10} \int \frac{(10a_0 z_1^4 + 15a_1 z_1^2 z_2^2 + 3a_2 z_2^3)(z dz)}{z_2^2 \sqrt{f(z_1, z_2)}}.$$

Similarly, in the two elliptic fields we find that to the system of normal integrals of the first and second kind,

$$(27) \quad \begin{aligned} u_1 &= \int \frac{(\zeta d\zeta)}{\sqrt{R_1(\zeta_1, \zeta_2)}} \\ u_2 &= - \int \frac{(a_0 \zeta_1 + a_1 \zeta_2)(\zeta d\zeta)}{\zeta_2 \sqrt{R_1(\zeta_1, \zeta_2)}}, \end{aligned}$$

1) Picard — Bull. d. l. Soc. Math. d. F., t. 11, pg. 43; Kowalevski — Acta Mathematica, Bd. 4, pg. 400.

2) Klein — Hyperelliptische Sigmafunctionen, Mathematische Annalen, Bd. 27, pg. 443.

3) Cf. Wiltheiss — Ueber eine partielle Differentialgleichung etc., Math. Annalen, Bd. 29, pg. 275; and Bolza — On Weierstrass' Systems of Hyper elliptic Integrals etc. (A paper read before the Mathematical Congress, Chicago, 1893).

belongs the covariant commutative integral of the third kind,

$$(28) \quad Q_{1\xi\eta}^{\xi'\eta'} = \int_{\eta}^{\xi} \int_{\eta'}^{\xi'} \frac{\sqrt{R_1(\zeta_1, \zeta_2)} \sqrt{R_1(\zeta'_1, \zeta'_2)} + F_1(\zeta_1, \zeta_2; \zeta'_1, \zeta'_2)}{2(\zeta'_2)^2} \frac{(\zeta d\zeta)(\zeta' d\zeta')}{\sqrt{R_1(\zeta_1, \zeta_2)} \sqrt{R_1(\zeta'_1, \zeta'_2)}},$$

and to the system,

$$(29) \quad \begin{aligned} \bar{u}_1 &= \int \frac{(\zeta d\zeta)}{\sqrt{R_2(\zeta_1, \zeta_2)}} \\ \bar{u}_2 &= - \int \frac{(2a_0 \zeta_1^2 + 3a_1 \zeta_1 \zeta_2 + a_2 \zeta_2^2)(\zeta d\zeta)}{\zeta_2^2 \sqrt{R_2(\zeta_1, \zeta_2)}}, \end{aligned}$$

belongs

$$(30) \quad Q_{2\xi\eta}^{\xi'\eta'} = \int_{\eta}^{\xi} \int_{\eta'}^{\xi'} \frac{\sqrt{R_2(\zeta_1, \zeta_2)} \sqrt{R_2(\zeta'_1, \zeta'_2)} + F_2(\zeta_1, \zeta_2; \zeta'_1, \zeta'_2)}{2(\zeta'_2)^2} \frac{(\zeta d\zeta)(\zeta' d\zeta')}{\sqrt{R_2(\zeta_1, \zeta_2)} \sqrt{R_2(\zeta'_1, \zeta'_2)}},$$

where $F_1(\zeta_1, \zeta_2; \zeta'_1, \zeta'_2)$ and $F_2(\zeta_1, \zeta_2; \zeta'_1, \zeta'_2)$ are the second polars of $R_1(\zeta_1, \zeta_2)$ and $R_2(\zeta_1, \zeta_2)$ respectively.

By inspection we are easily led to the result:

Theorem IV. *The hyperelliptic integrals w_1, w_2, w_3, w_4 admit of reduction to elliptic integrals as follows:*

$$(31) \quad \begin{aligned} w_1 &= u_1 \\ w_2 &= \bar{u}_2 \\ w_3 &= \frac{1}{2} u_2 + \frac{1}{5} a_1 u_1 \\ w_4 &= \frac{1}{2} \bar{u}_2 + \frac{1}{5} a_2 \bar{u}_1. \end{aligned}$$

If the periods of u_2 along the paths K, K' are $2\eta, 2\eta'$, and the periode of $\bar{u}_2$ along the paths $\bar{K}, \bar{K}'$ are $2\bar{\eta}, 2\bar{\eta}'$, then the integrals w_3, w_4 have the following table of integrals

	K_1	K_2	K_3	K_4
w_3	$2\eta + \frac{4}{5} a_1 \omega,$	0	$\eta' + \frac{2}{5} a_1 \omega',$	$\eta + \frac{2}{5} a_1 \omega$
w_4	0	$2\bar{\eta} + \frac{4}{5} a_2 \bar{\omega}$	$\bar{\eta}' + \frac{2}{5} a_2 \bar{\omega},$	$\eta' + \frac{2}{5} a_2 \omega'$

B. Integrals of the third kind. Passing to non-homogeneous

notation for convenience, and introducing the integrals

$$S = \int_y^x \int_{y'}^{x'} \frac{d}{dz'} \left(\frac{1}{2} \frac{\sqrt{f(z)} + \sqrt{f(z')}}{(z - z') \sqrt{f(z)}} \right) dz dz'$$

$$S_1 = \int_{\eta}^{\xi} \int_{\eta'}^{\xi'} \frac{d}{d\zeta'} \left(\frac{1}{2} \frac{\sqrt{R_1(\zeta)} + \sqrt{R_1(\zeta')}}{(\zeta - \zeta') \sqrt{R_1(\zeta)}} \right) d\zeta d\zeta'$$

$$S_2 = \int_{\eta}^{\xi} \int_{\eta'}^{\xi'} \frac{d}{d\zeta'} \left(\frac{1}{2} \frac{\sqrt{R_2(\zeta)} + \sqrt{R_2(\zeta')}}{(\zeta - \zeta') \sqrt{R_2(\zeta)}} \right) d\zeta d\zeta',$$

an easy calculation establishes the relation,

$$(32) \quad S_1 + S_2 = 2S + \log \frac{x + x'}{x + y'} \cdot \frac{y + y'}{y + x'} + 2m\pi i$$

where m is an integer depending for its value upon the paths of integration xy , $x'y'$.

Returning to homogeneous notation, the integrals

$$Q_{xy}^{x'y'}, \quad Q_{1\xi\eta}^{\xi'\eta'}, \quad Q_{2\xi\eta}^{\xi'\eta'}$$

are expressible in terms of S , S_1 , S_2 by the formulae¹⁾

$$Q_{xy}^{x'y'} = S - \sum_{\alpha=1,2} w_{\alpha}^{xy} w_{2+\alpha}^{x'y'}$$

$$Q_{1\xi\eta}^{\xi'\eta'} = S_1 - u_1^{\xi\eta} u_2^{\xi'\eta'}$$

$$Q_{2\xi\eta}^{\xi'\eta'} = S_2 - u_1^{\xi\eta} u_2^{\xi'\eta'}.$$

From this follows at once:

Theorem V. *The covariant commulative integral $Q_{xy}^{x'y'}$ satisfies the reduction-formula,*

$$(33) \quad \begin{cases} 2Q_{xy}^{x'y'} + \log \frac{x_1 x'_2 + x_2 x'_1}{x_1 y'_2 + x_2 y'_1} \cdot \frac{y_1 y'_2 + y_2 y'_1}{y_1 x'_2 + y_2 x'_1} + 2m\pi i = Q_{1\xi\eta}^{\xi'\eta'} + Q_{2\xi\eta}^{\xi'\eta'} \\ \quad - \frac{2}{\theta} (a_1 u_1^{\xi\eta} u_1^{\xi'\eta'} + a_2 u_1^{\xi\eta} u_1^{\xi'\eta'}). \end{cases}$$

1) Wiltheiss — Math. Annalen, Bd. 33, pg. 269—270; Bolza — On Weierstrass' Systems etc., pg. 11.

The hyperelliptic integrals of the second kind can now be readily expressed in terms of elliptic σ -functions.

For¹⁾

$$u_2^{\xi\eta} + \frac{\sqrt{R_1(\xi_1, \xi_2)} + \sqrt{R_1(\eta_1, \eta_2)}}{2(\xi\eta)} = \frac{\sigma'}{\sigma}(u_1^{\xi\eta})$$

$$\bar{u}_2^{\xi\eta} + \frac{\sqrt{R_2(\xi_1, \xi_2)} + \sqrt{R_2(\eta_1, \eta_2)}}{2(\xi\eta)} = \frac{\bar{\sigma}'}{\bar{\sigma}}(\bar{u}_1^{\xi\eta})$$

σ referring to g_2, g_3 , and $\bar{\sigma}$ referring to $\bar{g}_2, \bar{g}_3$.

Hence, from (31) follows

$$w_3^{xy} = \frac{1}{2} \cdot \frac{\sigma'}{\sigma}(u_1^{\xi\eta}) + \frac{a_1}{5} u_1^{\xi\eta} - \frac{\sqrt{R_1(\xi_1, \xi_2)} + \sqrt{R_1(\eta_1, \eta_2)}}{4(\xi\eta)}$$

$$w_4^{xy} = \frac{1}{2} \cdot \frac{\bar{\sigma}'}{\bar{\sigma}}(\bar{u}_1^{\xi\eta}) + \frac{a_2}{5} \bar{u}_1^{\xi\eta} - \frac{\sqrt{R_2(\xi_1, \xi_2)} + \sqrt{R_2(\eta_1, \eta_2)}}{4(\xi\eta)}.$$

Also, since²⁾

$$Q_{1\xi\eta}^{\xi'\eta'} = \log \frac{\sigma(u_1^{\xi\xi'}) \sigma(u_1^{\eta\eta'})}{\sigma(u_1^{\xi\eta'}) \sigma(u_1^{\eta\xi'})}$$

$$Q_{2\xi\eta}^{\xi'\eta'} = \log \frac{\bar{\sigma}(\bar{u}_1^{\xi\xi'}) \bar{\sigma}(\bar{u}_1^{\eta\eta'})}{\bar{\sigma}(\bar{u}_1^{\xi\eta'}) \bar{\sigma}(\bar{u}_1^{\eta\xi'})},$$

we obtain from (33)

$$(34) \left\{ \begin{aligned} & 2Q_{xy}^{x'y'} + \log \frac{x_1 x'_2 + x_2 x'_1}{x_1 y'_2 + x_2 y'_1} \cdot \frac{y_1 y'_2 + y_2 y'_1}{y_1 x'_2 + y_2 x'_1} = \log \frac{\sigma(u_1^{\xi\xi'}) \sigma(u_1^{\eta\eta'})}{\sigma(u_1^{\xi\eta'}) \sigma(u_1^{\eta\xi'})} \\ & + \log \frac{\bar{\sigma}(\bar{u}_1^{\xi\xi'}) \bar{\sigma}(\bar{u}_1^{\eta\eta'})}{\bar{\sigma}(\bar{u}_1^{\xi\eta'}) \bar{\sigma}(\bar{u}_1^{\eta\xi'})} - \frac{2}{5} (a_1 u_1^{\xi\eta} u_1^{\xi'\eta'} + a_2 \bar{u}_1^{\xi\eta} \bar{u}_1^{\xi'\eta'}) . \end{aligned} \right.$$

§ 6. The Hyperelliptic σ -Functions.

We now propose to express Klein's hyperelliptic σ -functions in terms of elliptic σ -functions, and for this purpose we make

1) See, for instance, Bolza — On the First and Second Logarithmic Derivatives of the Hyperelliptic σ -functions, American Journal, Vol. XVII, pg. 11.

2) Klein — Hyperelliptische Sigmafunctionen, Math. Ann. Bd. 27, pg. 456.

use of their definition in terms of the integral of the third kind¹⁾ $Q_{xy}^{\overline{xy}}$, in which the path of integration $\overline{xy}$ is the conjugate of the path xy , point for point. When we place this restriction on the path $x'y'$, formula (33) reduces to

$$(35) \quad 2Q_{xy}^{\overline{xy}} + \log \frac{4x_1 x_2 y_1 y_2}{(x_1 y_2 + x_2 y_1)^2} = Q_{1, \xi\eta}^{\overline{\xi\eta}} + Q_{2, \xi\eta}^{\overline{\xi\eta}} + \frac{2}{5} [a_1 (u_1^{\xi\eta})^2 + a_2 (\overline{u}_1^{\xi\eta})^2]$$

the paths of integration now being such that $m = 0$.

Since the σ -functions are covariants²⁾ of the sextic $f(z_1, z_2)$, we may perform our computations on the canonical form (7), or, as we may also write it,

$$f(z_1, z_2) = a_0(z_1^2 - \alpha^2 z_2^2)(z_1^2 - \beta^2 z_2^2)(z_1^2 - \gamma^2 z_2^2)$$

to which the original sextic has been reduced by a linear transformation of determinant 1.

§ 7. The Odd σ -Functions.

To every decomposition of the sextic into the product of a linear and a quintic factor corresponds an odd σ -function. If, for example $f = m \cdot n\phi\chi$, where $m = m_0(z_1 - \alpha z_2)$, $n = n_0(z_1 + \alpha z_2)$, and ϕ, χ have the same meaning as in (6), the corresponding function we shall denote by $\sigma_m(w_1, w_2)$, and, in accordance with Klein³⁾, it may be defined by the equation

$$\sigma_m(w_1, w_2) = \frac{(xy) \sqrt{m(x) \cdot m(y)}}{2 \sqrt[4]{f(x) \cdot f(y)}} \cdot e^{\frac{1}{2} Q_{xy}^{\overline{xy}}}$$

where

$$w_1 = \int_y^x \frac{z_1(zdz)}{\sqrt{f(z_1, z_2)}}, \quad w_2 = \int_y^x \frac{z_2(zdz)}{\sqrt{f(z_1, z_2)}}.$$

On the other hand, let $\sigma_1(u)$, $\sigma_2(u)$, $\sigma_3(u)$ be the three even σ -functions belonging to the invariants⁴⁾ g_2 , g_3 , and $\overline{\sigma}_1(\overline{u})$, $\overline{\sigma}_2(\overline{u})$, $\overline{\sigma}_3(\overline{u})$ those belonging to $\overline{g}_2$, $\overline{g}_3$, where

1) l. c., pag. 449.

2) l. c., pag. 439.

3) l. c., pag. 449.

4) Cf. § 2 of the preceding.

$$u = \int_{\eta}^{\xi} \frac{(\zeta d\zeta)}{\sqrt{R_1(\zeta_1, \zeta_2)}}, \quad \bar{u} = \int_{\eta}^{\xi} \frac{(\zeta d\zeta)}{\sqrt{R_2(\zeta_1, \zeta_2)}}.$$

Then, in accordance with Klein's definitions (l. c. pg. 455), and in view of formulae ¹⁾ (15), we write

$$(36) \quad \left\{ \begin{aligned} \sigma_1(u) &= \frac{\sqrt{4a_0\xi_2(\xi_1 - \alpha^2\xi_2)(\eta_1 - \beta^2\eta_2)(\eta_1 - \gamma^2\eta_2)} + \sqrt{4a_0\eta_2(\eta_1 - \alpha^2\eta_2)(\xi_1 - \beta^2\xi_2)(\xi_1 - \gamma^2\xi_2)}}{2\sqrt[4]{R_1(\xi)R_1(\eta)}} e^{\frac{1}{2}Q_1\frac{\xi\eta}{\xi_1\xi_2}} \\ \sigma_1(\bar{u}) &= \frac{\sqrt{4a_0\xi_1(\xi_1 - \alpha^2\xi_2)(\eta_1 - \beta^2\eta_2)(\eta_1 - \gamma^2\eta_2)} + \sqrt{4a_0\eta_1(\eta_1 - \alpha^2\eta_2)(\xi_1 - \beta^2\xi_2)(\xi_1 - \gamma^2\xi_2)}}{2\sqrt[4]{R_2(\xi)R_2(\eta)}} e^{\frac{1}{2}Q_2\frac{\xi\eta}{\xi_1\xi_2}} \end{aligned} \right.$$

while $\sigma_2(u), \sigma_2(\bar{u})$ are obtained directly from $\sigma_1(u), \sigma_1(\bar{u})$ by interchanging α and β , and $\sigma_3(u), \sigma_3(\bar{u})$ are similarly obtained by interchanging α and γ .

The odd elliptic σ -functions are

$$(37) \quad \left\{ \begin{aligned} \sigma(u) &= \frac{(\xi\eta)}{\sqrt[4]{R_1(\xi)R_1(\eta)}} e^{\frac{1}{2}Q_1\frac{\xi\eta}{\xi_1\xi_2}} \\ \bar{\sigma}(\bar{u}) &= \frac{(\xi\eta)}{\sqrt[4]{R_2(\xi)R_2(\eta)}} e^{\frac{1}{2}Q_2\frac{\xi\eta}{\xi_1\xi_2}} \end{aligned} \right.$$

It may be remarked, in passing, that the sign ambiguities in the above expressions need not be taken into account, since the functions in which they occur will hereafter be used only in combinations from which these ambiguities virtually disappear.

In order to express $\sigma_m(w_1, w_2)$ in terms of the elliptic σ -functions, we observe, in the first place, that $\sigma_m^2(w_1, w_2)$ is the product of an exponential $e^{\overline{Q}_{xy}}$, and an algebraic part whose denominator is $4\sqrt{f(x)f(y)}$ and whose numerator is rational in x_1, x_2, y_1, y_2 . Further, if each of the four products $\sigma_1(u)\bar{\sigma}_1(\bar{u}), \sigma_2(u)\bar{\sigma}_2(\bar{u}),$

1) If we denote the left hand of (36) by $\sigma_\lambda(u)$, and let $\sigma_\mu(u), \sigma_\nu(u)$ be obtained from $\sigma_\lambda(u)$ by the permutations $(\alpha\beta), (\alpha\gamma)$ respectively, then, since $\sigma_\lambda(u) = 1 - \frac{1}{2}e_\lambda u^2 + \dots$, by applying to this last equation, and also to (15), the permutations $(\alpha\beta), (\alpha\gamma)$ and comparing results, we find $\lambda = 1, \mu = 2, \nu = 3$.

$\sigma_s(u)\bar{\sigma}_s(\bar{u})$, $\sigma(u)\bar{\sigma}(\bar{u})$ be multiplied by the exponential

$$e^{\frac{1}{6}[a_1(u^{\xi\eta})^2 + a_2(\bar{u}^{\xi\eta})^2]},$$

which will be denoted briefly by $\varepsilon(u, \bar{u})$, the resulting quantities will each consist of the product of an algebraic part together with the exponential factor

$$(38) \quad \varepsilon(u, \bar{u}) \cdot e^{\frac{1}{2}Q_1 \frac{\bar{\xi\eta}}{\xi\eta} + \frac{1}{2}Q_2 \frac{\bar{\xi\eta}}{\xi\eta}}.$$

By means of (35) the exponential (38) can be replaced by

$$\frac{2\sqrt{x_1x_2y_1y_2}}{x_1y_2 + x_2y_1} \cdot e^{Q \frac{xy}{xy}}.$$

When we make the substitution

$$\frac{\zeta_1}{\zeta_2} = \frac{z_1^2}{z_2^2},$$

the four products $\varepsilon(u, \bar{u})\bar{\sigma}_1(u)\bar{\sigma}_1(\bar{u})$, etc., will each consist of the exponential factor

$$e^{Q \frac{xy}{xy}},$$

and an algebraic factor, rational in x_1, x_2, y_1, y_2 except for $\sqrt{f(x)f(y)}$ which occurs in the denominator. We see, on inspection, that $\sigma_m^2(w_1, w_2)$ cannot be expressed in terms of any one of the above four products alone. The next simplest expression being a linear combination, we proceed to determine whether constants A, B, C, D can be found such that

$$(39) \quad \sigma_m^2(w_1, w_2) = \varepsilon(u, \bar{u})[A\sigma_1(u)\bar{\sigma}_1(\bar{u}) + B\sigma_2(u)\bar{\sigma}_2(\bar{u}) + C\sigma_3(u)\bar{\sigma}_3(\bar{u}) + D\sigma(u)\bar{\sigma}(\bar{u})]$$

When we make the substitutions indicated above, divide out the common factors, and clear of fractions, we have the relation

$$(40) \quad \left\{ \begin{aligned} & (x_1y_2 + x_2y_1)(x_1y_2 - x_2y_1)^2 \cdot m_0^2(x_1 - \alpha x_2)(y_1 - \alpha y_2) \\ & = 2Aa_0[x_1x_2(x_1^2 - \alpha^2x_2^2)(y_1^2 - \beta^2y_2^2)(y_1^2 - \gamma^2y_2^2) + y_1y_2(y_1^2 - \alpha^2y_2^2)(x_1^2 - \beta^2x_2^2)(x_1^2 - \gamma^2x_2^2)] \\ & + 2Ba_0[x_1x_2(x_1^2 - \beta^2x_2^2)(y_1^2 - \alpha^2y_2^2)(y_1^2 - \gamma^2y_2^2) + y_1y_2(y_1^2 - \beta^2y_2^2)(x_1^2 - \alpha^2x_2^2)(x_1^2 - \gamma^2x_2^2)] \\ & + 2Ca_0[x_1x_2(x_1^2 - \gamma^2x_2^2)(y_1^2 - \beta^2y_2^2)(y_1^2 - \alpha^2y_2^2) + y_1y_2(y_1^2 - \gamma^2y_2^2)(x_1^2 - \beta^2x_2^2)(x_1^2 - \alpha^2x_2^2)] \\ & + 4(A + B + C)(x_1y_2 + x_2y_1)\sqrt{f(x)f(y)} + 2D(x_1^2y_2^2 - x_2^2y_1^2)^2 \end{aligned} \right.$$

which must be an identity for all values of x_1, x_2, y_1, y_2 .

In this equation we make the two specializations

$$\begin{array}{ll} x_1 = \alpha & x_1 = \gamma \\ 1) \quad y_1 = \beta & 2) \quad y_1 = \alpha \\ x_2 = y_2 = 1 & x_2 = y_2 = 1 \end{array}$$

and obtain the conditions

$$\begin{aligned} Aa_0\beta(\gamma^2 - \alpha^2) - Ba_0\alpha(\beta^2 - \gamma^2) + D &= 0 \\ Aa_0\gamma(\alpha^2 - \beta^2) - Ca_0\alpha(\beta^2 - \gamma^2) - D &= 0. \end{aligned}$$

If, further, we make $w_1 = w_2 = u = \bar{u} = 0$ in (39), we have the additional condition

$$A + B + C = 0.$$

These three equations of condition give for the ratios of the constants A, B, C, D the values

$$\frac{B}{D} = -\frac{C}{D} = \frac{1}{a_0\alpha(\beta^2 - \gamma^2)}, \quad A = 0.$$

By putting $x_1 = -\alpha$, $y_1 = \beta$, $x_2 = y_2 = 1$, D is found to have the value, $D = -\frac{1}{2}m_0^2\alpha$.

If, now, these values be substituted in (39), the relation is found to be, in fact, an identity. Accordingly we have

$$(41) \quad \sigma_m^2(w_1, w_2) = B\varepsilon(u, \bar{u})[\sigma_2(u)\sigma_2(\bar{u}) - \sigma_3(u)\sigma_3(\bar{u}) + \alpha_0\alpha(\beta^2 - \gamma^2)\sigma(u)\sigma(\bar{u})],$$

where

$$B = -\frac{m_0^3}{2a_0(\beta^2 - \gamma^2)}.$$

It only remains to express the coefficients of (41) in terms of the simultaneous invariants of m, n, ϕ, χ . For this purpose let

$$\begin{aligned} m &= m_s = m'_s = m_1z_1 + m_2z_2 \\ n &= n_s = n'_s = n_1z_1 + n_2z_2. \end{aligned}$$

Then, in the symbolic notation, we have

$$(42) \quad \left\{ \begin{array}{l} (mn) = 2m_0n_0\alpha \\ (\phi n)(\chi n')(\phi \chi) = -\alpha n_0^2\phi_0\chi_0(\beta^2 - \gamma^2) \end{array} \right.$$

from which we obtain

$$B = \frac{\frac{1}{4}(mn)}{(\psi n)(\chi n')(\psi \chi)}.$$

Also, from (12) and (42), remembering the notation of (15),

$$a_0 \alpha (\beta^2 - \gamma^2) = \frac{1}{2}(mn) E.$$

§ 8. The Even σ -Functions.

The even σ -functions are defined by the equation

$$\sigma_{\varphi_s, \psi_s}(w_1, w_2) = \frac{\sqrt{\varphi_s(x)\psi_s(y)} + \sqrt{\varphi_s(y)\psi_s(x)}}{2\sqrt{f(x)f(y)}} \cdot e^{\frac{1}{2}Q_{xy}^{xy}}$$

where φ_s, ψ_s are two cubics such that $\varphi_s \psi_s = f$. In the present special case *the ten functions divide themselves into two classes according to the way in which the linear factors of f are distributed in φ_s and ψ_s* . The linear factors of f may be denoted by m, n, p, q, r, s , where $mn = \varphi$, $pq = \psi$, $rs = \chi$. The two classes of σ -functions are, then,

I. φ_s, ψ_s of the form

$$\varphi_s(z_1, z_2) = \varphi(z_1, z_2) \cdot p(z_1, z_2)$$

$$\psi_s(z_1, z_2) = \chi(z_1, z_2) \cdot q(z_1, z_2).$$

The corresponding σ -functions may be designated by the notation

$$\sigma_{\begin{pmatrix} \varphi p \\ \chi q \end{pmatrix}}(w_1 w_2).$$

There are six such functions.

II. φ_s, ψ_s of the form

$$\varphi_s(z_1, z_2) = m(z_1, z_2) \cdot p(z_1, z_2) \cdot r(z_1, z_2)$$

$$\psi_s(z_1, z_2) = n(z_1, z_2) \cdot q(z_1, z_2) \cdot s(z_1, z_2).$$

The corresponding σ -functions will be denoted by $\sigma_{\begin{pmatrix} m p r \\ n q s \end{pmatrix}}(w_1, w_2)$.

They are four in number.

Class I. The six functions of this class must be expressible in the form

$$(43) \quad \left\{ \begin{aligned} \sigma_{\left(\begin{smallmatrix} \varphi p \\ \chi q \end{smallmatrix}\right)}(w_1, w_2) &= \varepsilon(u, \bar{u}) [A \sigma_1(u) \bar{\sigma}_1(\bar{u}) + B \sigma_2(u) \bar{\sigma}_2(\bar{u}) \\ &\quad + C \sigma_3(u) \bar{\sigma}_3(\bar{u}) + D \sigma(u) \bar{\sigma}(\bar{u})]. \end{aligned} \right.$$

For, by the addition, in equation (41), of a properly chosen set of half periods to w_1, w_2 , and the corresponding half periods to $u, \bar{u}$, the equation may be made to take the form of (43). This process would, in fact, lead us directly to the values of A, B, C, D , but, in the absence of sufficiently explicit formulae for effecting such a transformation, it is easier to determine these values by the method of indeterminate coefficients.

Substituting the explicit expressions for the σ -functions dividing out common factors, and clearing of fractions, we have

$$(44) \quad \left\{ \begin{aligned} &\frac{x_1 y_2 + x_2 y_1}{2} [a_0 (x_1^2 - \alpha^2 x_2^2) (x_1 - \beta x_2) (y_1^2 - \gamma^2 y_2^2) (y_1 + \beta y_2) + \\ &\quad + a_0 (y_1^2 - \alpha^2 y_2^2) (y_1 - \beta y_2) (x_1^2 - \gamma^2 x_2^2) (x_1 + \beta x_2) + 2\sqrt{f(x)f(y)}] \\ &= A a_0 [x_1 x_2 (x_1^2 - \alpha^2 x_2^2) (y_1^2 - \beta^2 y_2^2) (y_1^2 - \gamma^2 y_2^2) + y_1 y_2 (y_1^2 - \alpha^2 y_2^2) (x_1^2 - \beta^2 x_2^2) (x_1^2 - \gamma^2 x_2^2)] \\ &\quad + B a_0 [x_1 x_2 (x_1^2 - \beta^2 x_2^2) (y_1^2 - \alpha^2 y_2^2) (y_1^2 - \gamma^2 y_2^2) + y_1 y_2 (y_1^2 - \beta^2 y_2^2) (x_1^2 - \alpha^2 x_2^2) (x_1^2 - \gamma^2 x_2^2)] \\ &\quad + C a_0 [x_1 x_2 (x_1^2 - \gamma^2 x_2^2) (y_1^2 - \beta^2 y_2^2) (y_1^2 - \alpha^2 y_2^2) + y_1 y_2 (y_1^2 - \gamma^2 y_2^2) (x_1^2 - \beta^2 x_2^2) (x_1^2 - \alpha^2 x_2^2)] \\ &\quad + (A + B + C) (x_1 y_2 + x_2 y_1) \sqrt{f(x)f(y)} + D (x_1^2 y_2^2 - x_2^2 y_1^2)^2. \end{aligned} \right.$$

Making the four specializations,

$$1) \begin{cases} x_1 = \alpha \\ y_1 = \beta \\ x_2 = y_2 = 1 \end{cases} \quad 2) \begin{cases} x_1 = -\alpha \\ y_1 = \beta \\ x_2 = y_2 = 1 \end{cases} \quad 3) \begin{cases} x_1 = \gamma \\ y_1 = -\beta \\ x_2 = y_2 = 1 \end{cases} \quad 4) \begin{cases} w_1 = w_2 = 0 \\ u = \bar{u} = 0 \end{cases}$$

we obtain the following conditions,

$$\begin{aligned} A a_0 \beta (\gamma^2 - \alpha^2) - B a_0 \alpha (\beta^2 - \gamma^2) + D &= 0 \\ A a_0 \beta (\gamma^2 - \alpha^2) + B a_0 \alpha (\beta^2 - \gamma^2) + D &= 0 \\ B a_0 \gamma (\alpha^2 - \beta^2) + C a_0 \beta (\gamma^2 - \alpha^2) + D &= 0 \\ A + B + C &= 1, \end{aligned}$$

from which we find

$$B = 0, \quad A = C = \frac{1}{2}, \quad D = -\frac{1}{2} a_0 \beta (\gamma^2 - \alpha^2) = -\frac{1}{2} (pq) F.$$

Class II. The four functions of this class are expressible in a form similar to (43). By the same method as in the prece-

ding, we derive the equation

$$(45) \left\{ \begin{aligned} & \frac{x_1 y_2 + x_2 y_1}{2} [a_0 (x_1 - \alpha x_2) (x_1 - \beta x_2) (x_1 - \gamma x_2) (y_1 + \alpha y_2) (y_1 + \beta y_2) (y_1 + \gamma y_2) \\ & + a_0 (y_1 - \alpha y_2) (y_1 - \beta y_2) (y_1 - \gamma y_2) (x_1 + \alpha x_2) (x_1 + \beta x_2) (x_1 + \gamma x_2) + 2\sqrt{f(x)f(y)}] \\ & = A a_0 [x_1 x_2 (x_1^2 - \alpha^2 x_2^2) (y_1^2 - \beta^2 y_2^2) (y_1^2 - \gamma^2 y_2^2) + y_1 y_2 (y_1^2 - \alpha^2 y_2^2) (x_1^2 - \beta^2 x_2^2) (x_1^2 - \gamma^2 x_2^2)] \\ & + B a_0 [x_1 x_2 (x_1^2 - \beta^2 x_2^2) (y_1^2 - \alpha^2 y_2^2) (y_1^2 - \gamma^2 y_2^2) + y_1 y_2 (y_1^2 - \beta^2 y_2^2) (x_1^2 - \alpha^2 x_2^2) (x_1^2 - \gamma^2 x_2^2)] \\ & + C a_0 [x_1 x_2 (x_1^2 - \gamma^2 x_2^2) (y_1^2 - \beta^2 y_2^2) (y_1^2 - \alpha^2 y_2^2) + y_1 y_2 (y_1^2 - \gamma^2 y_2^2) (x_1^2 - \beta^2 x_2^2) (x_1^2 - \alpha^2 x_2^2)] \\ & + (A + B + C) (x_1 y_2 + x_2 y_1) \sqrt{f(x)f(y)} + D (x_1^2 y_2^2 - x_2^2 y_1^2)^2. \end{aligned} \right.$$

By making the following specializations,

$$1) \begin{cases} x_1 = \alpha \\ y_1 = \beta \\ x_2 = y_2 = 1 \end{cases} \quad 2) \begin{cases} x_1 = \alpha \\ y_1 = \gamma \\ x_2 = y_2 = 1 \end{cases} \quad 3) \begin{cases} x_1 = \beta \\ y_1 = \gamma \\ x_2 = y_2 = 1 \end{cases} \quad 4) \begin{cases} w_1 = w_2 = 0 \\ u = \bar{u} = 0 \end{cases}$$

we obtain the conditions,

$$A a_0 \beta (\gamma^2 - \alpha^2) - B a_0 \alpha (\beta^2 - \gamma^2) + D = 0$$

$$A a_0 \gamma (\alpha^2 - \beta^2) - C a_0 \alpha (\beta^2 - \gamma^2) - D = 0$$

$$B a_0 \gamma (\alpha^2 - \beta^2) - C a_0 \beta (\gamma^2 - \alpha^2) + D = 0$$

$$A + B + C = 1$$

from which we find

$$A = \frac{\alpha (\beta^2 - \gamma^2)}{(\alpha - \beta) (\beta - \gamma) (\gamma - \alpha)}$$

$$B = \frac{\beta (\gamma^2 - \alpha^2)}{(\alpha - \beta) (\beta - \gamma) (\gamma - \alpha)}$$

$$C = \frac{\gamma (\alpha^2 - \beta^2)}{(\alpha - \beta) (\beta - \gamma) (\gamma - \alpha)}$$

$$D = 0.$$

These constants are readily expressed in the invariant form when we observe that

$$a_0 \alpha (\beta^2 - \gamma^2) = (mn) E$$

$$a_0 \beta (\gamma^2 - \alpha^2) = (pq) F$$

$$a_0 \gamma (\alpha^2 - \beta^2) = (rs) G$$

$$a_0 (\alpha - \beta) (\beta - \gamma) (\gamma - \alpha) = a_0 \alpha (\beta^2 - \gamma^2) + a_0 \beta (\gamma^2 - \alpha^2) + a_0 \gamma (\alpha^2 - \beta^2)$$

$$= (mn) E + (pq) F + (rs) G.$$

The results of the present and the preceding section we will summarize, for convenience, in the following theorem.

Theorem VI. Let $\varphi = mn$, $\phi = pq$, $\chi = rs$ be three quadratic forms in the variables z_1, z_2 , whose skew invariant R is zero, and let $m, n; p, q; r, s$ be their linear factors. Further, let $A_1, A_2; A'_1, A'_2$ be constants such that φ, ϕ, χ are each linearly expressible in terms of $(A_1 z_1 + A_2 z_2)^2, (A'_1 z_1 + A'_2 z_2)^2$. If, besides, we put

$$w_1 = \int_y^x \frac{(A_1 z_1 + A_2 z_2)(z dz)}{\sqrt{\varphi \phi \chi}}, \quad w_2 = \int_y^x \frac{(A'_1 z_1 + A'_2 z_2)(z dz)}{\sqrt{\varphi \phi \chi}}$$

and suppose $e_1, e_2, e_3; \bar{e}_1, \bar{e}_2, \bar{e}_3; a_1, a_2; E, F, G$ to be defined by the equations (13) and (15), then each of the 16 hyperelliptic σ -functions belonging to $\sqrt{\varphi \phi \chi}$ is expressible in terms of two sets of elliptic σ -functions, belonging to e_1, e_2, e_3 and $\bar{e}_1, \bar{e}_2, \bar{e}_3$ respectively, by means of a formula coming under one of the three types:

$$\begin{aligned} & \sigma_m^2(w_1, w_2) = \\ &= \frac{(mn)}{4(\phi n)(\chi n')(\phi \chi)} e^{\frac{1}{2}(a_1 w_1^2 + a_2 w_2^2)} [\sigma_2(w_1) \bar{\sigma}_2(w_2) - \sigma_3(w_1) \bar{\sigma}_3(w_2) + \frac{1}{2}(mn) E \sigma(w_1) \bar{\sigma}(w_2)] \\ & \sigma_{\begin{pmatrix} \varphi p \\ \chi q \end{pmatrix}}^2(w_1, w_2) = \\ &= \frac{1}{2} e^{\frac{1}{2}(a_1 w_1^2 + a_2 w_2^2)} [\sigma_1(w_1) \bar{\sigma}_1(w_2) + \sigma_3(w_1) \bar{\sigma}_3(w_2) - (pq) F \sigma(w_1) \bar{\sigma}(w_2)] \\ & \sigma_{\begin{pmatrix} mpr \\ nqs \end{pmatrix}}^2(w_1, w_2) = \\ &= \frac{e^{\frac{1}{2}(a_1 w_1^2 + a_2 w_2^2)}}{(mn) E + (pq) F + (rs) G} [(mn) E \sigma_1(w_1) \bar{\sigma}_1(w_2) + (pq) F \sigma_2(w_1) \bar{\sigma}_2(w_2) + (rs) G \sigma_3(w_1) \bar{\sigma}_3(w_2)]. \end{aligned}$$

§ 8. The Kummer Surface.

A. Weber's Equation. Weber has shown that the $\mathfrak{F}$ -relation discovered by him, and proved to be identical with Kummer's equation of the 16-nodal quartic surface, represents, in the special case before us, the Fresnel's Wave Surface¹⁾. This can easily be shown, also, for the corresponding σ -relation.

For our present purposes it is convenient to make a slight modification of the notation used in §§ 6—7. Let $f(z_1, z_2)$ be

1) H. Weber — Crelle Bd. 84. pg. 353 (1878).

written as the product of its linear factors as follows,

$$f(z_1, z_2) = (z\alpha)(z\beta)(z\gamma)(z\alpha')(z\beta')(z\gamma').$$

Then, omitting the arguments of the σ -functions for the sake of brevity, we will denote by σ_α the function corresponding to the decomposition of the sextic into the linear factor $(z\alpha)$ and the remaining quintic, and by $\sigma\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix}\right)$ the function corresponding to

$$\varphi_3 = (z\alpha)(z\beta)(z\gamma), \quad \psi_3 = (z\alpha')(z\beta')(z\gamma').$$

The Weber equation, in its general form, will be a relation between the functions

$$p = \sigma_\alpha^2, \quad q = \sigma_\beta^2, \quad r = \sigma_\gamma^2, \quad s = \sigma\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix}\right).$$

From the relation¹⁾

$$(\beta\gamma)\sigma_\alpha\sigma\left(\begin{smallmatrix} \beta & \gamma & \alpha' \\ \beta' & \gamma' & \alpha \end{smallmatrix}\right) + (\gamma\alpha)\sigma_\beta\sigma\left(\begin{smallmatrix} \alpha & \alpha' & \gamma \\ \beta & \beta' & \gamma' \end{smallmatrix}\right) + (\alpha\beta)\sigma_\gamma\sigma\left(\begin{smallmatrix} \alpha & \alpha' & \beta \\ \gamma & \gamma' & \beta' \end{smallmatrix}\right) = 0$$

we eliminate the even functions by means of the formulae²⁾

$$(46) \quad \begin{cases} (\beta\gamma)\sigma\left(\begin{smallmatrix} \beta & \gamma & \alpha' \\ \beta' & \gamma' & \alpha \end{smallmatrix}\right) = (\beta\gamma)s + (\alpha\alpha')(\gamma\beta')(\gamma\gamma')q - (\alpha\alpha')(\beta\beta')(\beta\gamma')r \\ (\gamma\alpha)\sigma\left(\begin{smallmatrix} \alpha & \alpha' & \gamma \\ \beta & \beta' & \gamma' \end{smallmatrix}\right) = (\gamma\alpha)s - (\beta\alpha')(\gamma\beta')(\gamma\gamma')p + (\beta\alpha')(\alpha\beta')(\alpha\gamma')r \\ (\alpha\beta)\sigma\left(\begin{smallmatrix} \alpha & \alpha' & \beta \\ \gamma & \gamma' & \beta' \end{smallmatrix}\right) = (\alpha\beta)s - (\gamma\alpha')(\alpha\beta')(\alpha\gamma')q + (\gamma\alpha')(\beta\beta')(\beta\gamma')r \end{cases}$$

and obtain immediately the relation sought, viz.,

$$(47) \quad \begin{cases} \sqrt{(\beta\gamma)p[(\beta\gamma)s + (\alpha\alpha')(\gamma\beta')(\gamma\gamma')q - (\alpha\alpha')(\beta\beta')(\beta\gamma')r]} \\ + \sqrt{(\gamma\alpha)q[(\gamma\alpha)s - (\beta\alpha')(\gamma\beta')(\gamma\gamma')p + (\beta\alpha')(\alpha\beta')(\alpha\gamma')r]} \\ + \sqrt{(\alpha\beta)r[(\alpha\beta)s - (\gamma\alpha')(\alpha\beta')(\alpha\gamma')q + (\gamma\alpha')(\beta\beta')(\beta\gamma')r]} = 0. \end{cases}$$

If we compare this with Kummer's equation³⁾

1) Burkhardt — Systematik der hyperelliptischen Functionen, Math. Annalen, Bd. 35, pg. 242 (1890).

2) l. c. pg. 241.

3) Kummer — Ueber die Flächen vierten Grades, mit sechzehn singulären Punkten. Monatsberichte der Berl. Akad., 1864.

$\sqrt{p(\beta q + \gamma r + \delta s)} + \sqrt{q(\alpha' p + \gamma' r + \delta' s)} + \sqrt{r(\alpha'' p + \beta'' q + \delta'' s)} = 0$
 it is found that the constants satisfy the required conditions

$$\begin{aligned}\alpha' \gamma + \alpha'' \beta - \beta \gamma &= 0 \\ \alpha'' \gamma' + \beta'' \gamma - \alpha'' \beta'' &= 0.\end{aligned}$$

Referring to the reduction formulae (§§ 6–7), we observe that a new set of variables x_1, x_2, x_3, x_4 may be introduced by means of a linear transformation such that

$$\begin{aligned}x_1 &= \sigma_\lambda(u) \overline{\sigma_\lambda(u)} \\ x_2 &= \sigma_\mu(u) \overline{\sigma_\mu(u)} \\ x_3 &= \sigma_\nu(u) \overline{\sigma_\nu(u)} \\ x_4 &= \sigma(u) \overline{\sigma_\mu(u)}.\end{aligned}$$

The resulting equation is that of the Wave Surface¹⁾.

B. Borchardt's Equation. Borchardt's equation expresses the relation between the four functions of a Göpel's quadruple, such as, for example,

$$\sigma\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix}\right), \sigma\left(\begin{smallmatrix} \alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma \end{smallmatrix}\right), \sigma\left(\begin{smallmatrix} \alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma \end{smallmatrix}\right), \sigma\left(\begin{smallmatrix} \alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma \end{smallmatrix}\right).$$

Starting with the first equation of (46), we derive two others by permuting: 1st, α with β , and α' with β' ; 2nd, α with γ , and α' with γ' . Solving these three equations with respect to σ_α^2 , we obtain

$$(48) \left\{ \begin{aligned} &(\beta\beta')(\gamma\gamma')[(\beta\gamma')(\gamma\alpha')(\alpha\beta') + (\beta'\gamma)(\gamma'\alpha)(\alpha'\beta)]\sigma_\alpha^2 = (\beta'\gamma')(\gamma\alpha)(\alpha\beta)\sigma^2\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix}\right) \\ &-(\beta\gamma')(\gamma\alpha)(\alpha\beta')\sigma^2\left(\begin{smallmatrix} \alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma \end{smallmatrix}\right) - (\beta'\gamma)(\gamma'\alpha)(\alpha\beta)\sigma^2\left(\begin{smallmatrix} \alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma \end{smallmatrix}\right) + (\beta\gamma)(\gamma'\alpha)(\alpha\beta')\sigma^2\left(\begin{smallmatrix} \alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma \end{smallmatrix}\right) \end{aligned} \right.$$

and from this, by permuting α with α' , β with β' , and γ with γ' , we have

$$(48') \left\{ \begin{aligned} &(\beta\beta')(\gamma\gamma')[(\beta\gamma')(\gamma\alpha')(\alpha\beta') + (\beta'\gamma)(\gamma'\alpha)(\alpha'\beta)]\sigma_{\alpha'}^2 = (\beta\gamma)(\gamma'\alpha')(\alpha'\beta')\sigma^2\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix}\right) \\ &-(\beta'\gamma)(\gamma'\alpha')(\alpha'\beta)\sigma^2\left(\begin{smallmatrix} \alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma \end{smallmatrix}\right) - (\beta\gamma)(\gamma'\alpha)(\alpha'\beta')\sigma^2\left(\begin{smallmatrix} \alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma \end{smallmatrix}\right) + (\beta'\gamma)(\gamma'\alpha)(\alpha'\beta)\sigma^2\left(\begin{smallmatrix} \alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma \end{smallmatrix}\right) \end{aligned} \right.$$

1) Study — Sphärische Trigonometrie, orthogonale Substitutionen und elliptische Functionen, pg. 225.

Burkhardt's formula (94), (l. c. pag. 242), gives us

$$(\beta\beta')(\gamma\gamma')\sigma_{\alpha}\sigma_{\alpha'} = \sigma\left(\begin{smallmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{smallmatrix}\right)\sigma\left(\begin{smallmatrix}\alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma\end{smallmatrix}\right) - \sigma\left(\begin{smallmatrix}\alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma'\end{smallmatrix}\right)\sigma\left(\begin{smallmatrix}\alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma\end{smallmatrix}\right).$$

Squaring each member and eliminating $\sigma_{\alpha}^2, \sigma_{\alpha'}^2$, by means of (48) and (48'), we obtain, after a little reduction, the equation sought, viz.,

$$(49) \left\{ \begin{aligned} & (\beta\gamma)(\gamma'\alpha)(\alpha\beta)(\beta'\gamma')(\gamma'\alpha')(\alpha'\beta')\sigma^4\left(\begin{smallmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{smallmatrix}\right) + (\beta'\gamma)(\gamma'\alpha)(\alpha\beta')(\beta\gamma')(\gamma'\alpha')(\alpha'\beta)\sigma^4\left(\begin{smallmatrix}\alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma\end{smallmatrix}\right) \\ & + (\beta\gamma')(\gamma'\alpha)(\alpha\beta)(\beta'\gamma)(\gamma'\alpha')(\alpha'\beta')\sigma^4\left(\begin{smallmatrix}\alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma\end{smallmatrix}\right) + (\beta'\gamma')(\gamma'\alpha)(\alpha\beta')(\beta\gamma)(\gamma'\alpha')(\alpha'\beta)\sigma^4\left(\begin{smallmatrix}\alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma'\end{smallmatrix}\right) \\ & + 2 [(\beta\gamma')(\gamma'\alpha)(\alpha\beta') + (\beta'\gamma)(\gamma'\alpha)(\alpha'\beta)]\sigma^2\left(\begin{smallmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{smallmatrix}\right)\sigma^2\left(\begin{smallmatrix}\alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma'\end{smallmatrix}\right)\sigma^2\left(\begin{smallmatrix}\alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma\end{smallmatrix}\right)\sigma^2\left(\begin{smallmatrix}\alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma\end{smallmatrix}\right) \\ & - [(\gamma'\alpha')(\gamma'\alpha')(\alpha\beta)(\alpha\beta') + (\gamma'\alpha)(\gamma'\alpha)(\alpha'\beta)(\alpha'\beta')] [(\beta\gamma)(\beta'\gamma')\sigma^2\left(\begin{smallmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{smallmatrix}\right)\sigma^2\left(\begin{smallmatrix}\alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma'\end{smallmatrix}\right) \\ & \quad + (\beta'\gamma)(\beta'\gamma')\sigma^2\left(\begin{smallmatrix}\alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma'\end{smallmatrix}\right)\sigma^2\left(\begin{smallmatrix}\alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma\end{smallmatrix}\right)] \\ & + [(\alpha\beta')(\alpha'\beta')(\beta\gamma)(\beta'\gamma') + (\alpha\beta)(\alpha'\beta)(\beta'\gamma)(\beta'\gamma')] [(\gamma\alpha)(\gamma'\alpha')\sigma^2\left(\begin{smallmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{smallmatrix}\right)\sigma^2\left(\begin{smallmatrix}\alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma'\end{smallmatrix}\right) \\ & \quad + (\gamma'\alpha)(\gamma'\alpha')\sigma^2\left(\begin{smallmatrix}\alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma\end{smallmatrix}\right)\sigma^2\left(\begin{smallmatrix}\alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma\end{smallmatrix}\right)] \\ & + [(\beta\gamma')(\beta'\gamma')(\gamma\alpha)(\gamma'\alpha') + (\beta\gamma)(\beta'\gamma)(\gamma'\alpha)(\gamma'\alpha')] [(\alpha\beta)(\alpha'\beta')\sigma^2\left(\begin{smallmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{smallmatrix}\right)\sigma^2\left(\begin{smallmatrix}\alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma\end{smallmatrix}\right) \\ & \quad + (\alpha'\beta)(\alpha'\beta')\sigma^2\left(\begin{smallmatrix}\alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma'\end{smallmatrix}\right)\sigma^2\left(\begin{smallmatrix}\alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma\end{smallmatrix}\right)] = 0. \end{aligned} \right.$$

In the special case in which $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ are pairs in an involution, the 15 Göpel quadruples composed of even functions divide themselves into three classes.

Class I. One quadruple,

$$\sigma\left(\begin{smallmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{smallmatrix}\right), \sigma\left(\begin{smallmatrix}\alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma'\end{smallmatrix}\right), \sigma\left(\begin{smallmatrix}\alpha & \beta & \gamma' \\ \alpha' & \beta' & \gamma\end{smallmatrix}\right), \sigma\left(\begin{smallmatrix}\alpha & \beta' & \gamma' \\ \alpha' & \beta & \gamma\end{smallmatrix}\right).$$

Class II. Six quadruples of the type

$$\sigma\left(\begin{smallmatrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{smallmatrix}\right), \sigma\left(\begin{smallmatrix}\alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma'\end{smallmatrix}\right), \sigma\left(\begin{smallmatrix}\alpha & \alpha' & \beta \\ \gamma & \gamma' & \beta'\end{smallmatrix}\right), \sigma\left(\begin{smallmatrix}\alpha & \alpha' & \beta' \\ \gamma & \gamma' & \beta\end{smallmatrix}\right).$$

Class III. Eight quadruples of the type

$$\sigma \begin{pmatrix} \alpha \beta \gamma \\ \alpha' \beta' \gamma' \end{pmatrix}, \sigma \begin{pmatrix} \beta \beta' \alpha \\ \gamma \gamma' \alpha' \end{pmatrix}, \sigma \begin{pmatrix} \alpha \alpha' \gamma \\ \beta \beta' \gamma' \end{pmatrix}, \sigma \begin{pmatrix} \gamma \gamma' \beta \\ \alpha \alpha' \beta' \end{pmatrix}.$$

I. In the equation of Class I (see formula (49)) one of the terms disappears, since

$$(\beta \gamma') (\gamma \alpha') (\alpha \beta') + (\beta' \gamma) (\gamma' \alpha) (\alpha' \beta) = 0,$$

as we may immediately see where we take the sextic in the canonical form of § 6. Equation (49) then easily reduces to

$$(x_1^2 - x_2^2 - x_3^2 + x_4^2)^2 = 0$$

where

$$x_1^2 = \sqrt{(\beta \gamma) (\gamma \alpha) (\alpha \beta) (\beta' \gamma') (\gamma' \alpha') (\alpha' \beta')} \sigma^2 \begin{pmatrix} \alpha \beta \gamma \\ \alpha' \beta' \gamma' \end{pmatrix} = \Delta_1 \sigma^2 \begin{pmatrix} \alpha \beta \gamma \\ \alpha' \beta' \gamma' \end{pmatrix}$$

$$x_2^2 = \sqrt{(\beta' \gamma) (\gamma \alpha) (\alpha \beta') (\beta \gamma') (\gamma' \alpha') (\alpha' \beta)} \sigma^2 \begin{pmatrix} \alpha \beta' \gamma \\ \alpha' \beta \gamma' \end{pmatrix} = \Delta_2 \sigma^2 \begin{pmatrix} \alpha \beta' \gamma \\ \alpha' \beta \gamma' \end{pmatrix}$$

$$x_3^2 = \sqrt{(\beta \gamma') (\gamma' \alpha) (\alpha \beta) (\beta' \gamma) (\gamma \alpha') (\alpha' \beta')} \sigma^2 \begin{pmatrix} \alpha \beta \gamma' \\ \alpha' \beta' \gamma \end{pmatrix} = \Delta_3 \sigma^2 \begin{pmatrix} \alpha \beta \gamma' \\ \alpha' \beta' \gamma \end{pmatrix}$$

$$x_4^2 = \sqrt{(\beta' \gamma') (\gamma' \alpha) (\alpha \beta') (\beta \gamma) (\gamma \alpha') (\alpha' \beta)} \sigma^2 \begin{pmatrix} \alpha \beta' \gamma' \\ \alpha' \beta \gamma \end{pmatrix} = \Delta_4 \sigma^2 \begin{pmatrix} \alpha \beta' \gamma' \\ \alpha' \beta \gamma \end{pmatrix}$$

It is to be observed that the sign of one of the radicals may be chosen arbitrarily. The signs of the others are there determined by the conditions

$$\Delta_2 = -\Delta_1 \cdot (\gamma' \beta \alpha \beta')$$

$$\Delta_3 = -\Delta_1 \cdot (\gamma \alpha' \gamma' \beta)$$

$$\Delta_4 = -\Delta_1 \cdot (\beta' \alpha \gamma \alpha')$$

where, for brevity, $(\gamma' \beta \alpha \beta')$ is used to denote the anharmonic ratio $\frac{(\gamma' \beta)}{(\alpha \beta)} : \frac{(\gamma' \beta')}{(\alpha \beta')}$.

The Kummer surface, in this case, becomes a doubly-covered quadric surface.

II. Using the quadruple given for illustration under Class II, the equation connecting these functions may be obtained at once from (49) by permuting α' and γ' . If in this we make the substitution

$$x_1 = \sqrt[4]{(\beta\gamma)(\gamma\alpha)(\alpha\beta)(\beta'\gamma')(\gamma'\alpha')(\alpha'\beta')} \sigma \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix} = \sqrt{\Delta_1} \sigma \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}$$

$$x_2 = \sqrt[4]{(\beta'\gamma)(\gamma\alpha)(\alpha\beta)(\beta\gamma')(\gamma'\alpha')(\alpha'\beta')} \sigma \begin{pmatrix} \alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma' \end{pmatrix} = \sqrt{\Delta_2} \sigma \begin{pmatrix} \alpha & \beta' & \gamma \\ \alpha' & \beta & \gamma' \end{pmatrix}$$

$$x_3 = \sqrt[4]{(\beta\alpha')(\alpha'\alpha)(\alpha\beta)(\beta'\gamma)(\gamma\gamma')(\beta'\gamma')} \sigma \begin{pmatrix} \beta & \alpha & \alpha' \\ \beta' & \gamma & \gamma' \end{pmatrix} = \sqrt{\Delta_3} \sigma \begin{pmatrix} \beta & \alpha & \alpha' \\ \beta' & \gamma & \gamma' \end{pmatrix}$$

$$x_4 = \sqrt[4]{(\beta'\alpha')(\alpha'\alpha)(\alpha\beta)(\beta\gamma)(\gamma\gamma')(\beta'\gamma')} \sigma \begin{pmatrix} \beta' & \alpha & \alpha' \\ \beta & \gamma & \gamma' \end{pmatrix} = \sqrt{\Delta_4} \sigma \begin{pmatrix} \beta' & \alpha & \alpha' \\ \beta & \gamma & \gamma' \end{pmatrix}$$

the resulting equation takes the form

$$\begin{aligned} x_1^4 + x_2^4 + x_3^4 + x_4^4 + \lambda x_1 x_2 x_3 x_4 + \mu (x_1^2 x_2^2 + x_3^2 x_4^2) \\ + \nu (x_1^2 x_3^2 + x_2^2 x_4^2 + x_1^2 x_4^2 + x_2^2 x_3^2) = 0. \end{aligned}$$

The roots of unity involved in the above radicals we may choose arbitrarily, subject only to the conditions

$$\begin{aligned} \Delta_4 &= -\Delta_3 \\ \Delta_2 &= -\Delta_1(\beta\alpha'\beta'\gamma). \end{aligned}$$

When we make the linear transformation

$$\begin{aligned} x_1 &= y_1 + y_2 \\ x_2 &= y_3 + y_4 \\ x_3 &= y_1 - y_2 \\ x_4 &= y_3 - y_4 \end{aligned}$$

the equation assumes the simpler form

$$\begin{aligned} y_1^4 + y_2^4 + y_3^4 + y_4^4 + \lambda' (y_1^2 y_2^2 + y_3^2 y_4^2) \\ + \mu' (y_1^2 y_3^2 + y_2^2 y_4^2) + \nu' (y_1^2 y_4^2 + y_2^2 y_3^2) = 0. \end{aligned}$$

This reduces to the well known equation of the Wave Surface

$$\frac{a^2 x^2}{a^2 - r^2} + \frac{b^2 y^2}{b^2 - r^2} + \frac{c^2 z^2}{c^2 - r^2} = 0$$

when we make the substitutions

$$\begin{aligned} \lambda' &= \frac{b^2 + c^2}{bc}, \quad \mu' = \frac{c^2 + a^2}{ca}, \quad \nu' = \frac{a^2 + b^2}{ab}; \\ x^2 &= -\frac{bc y_2^2}{y_1^2}, \quad y^2 = -\frac{ca y_3^2}{y_1^2}, \quad z^2 = -\frac{ab y_4^2}{y_1^2}; \\ r^2 &= x^2 + y^2 + z^2. \end{aligned}$$

The equations of Class III represent a surface more properly associated with the reduction problem for a transformation of the fourth degree, and will not be considered further in this connection. It is what Humbert calls an Elliptic Kummer Surface of index 4¹).

1) G. Humbert — Sur les surfaces de Kummer elliptiques. American Journal, Vol. XVI, pg. 221.

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V i t a.

I, John Irwin Hutchinson, was born at Bangor, Me., on the 12th of April, 1867. I completed my preparatory course at the Edward Little High School, Auburn, in 1885, and in 1889 was graduated from Bates College, Lewiston, with the degree, Bachelor of Arts. During the two years 1890—92, I was Scholar, and Fellow in Mathematics at Clark University, where I attended lectures by Bolza, Perott, Story, Taber, and White. The two following years I was Fellow in Mathematics at the University of Chicago, attending lectures by Bolza, Hale, Laves, Maschke, Moore, and Young. Also for a considerable portion of the time during one year I was occupied in laboratory experiments and astronomical observations at the Kenwood Astrophysical Observatory of the University of Chicago.

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